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Problems of geometric non-linearity and stability in the mechanics of thin shells and rectilinear columns☆

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Abstract

An analysis of the current state of the geometrically non-linear theory of elasticity and of thin shells is presented in the case of small deformations but large displacements and rotations, the ratios of which are known as the ratios of the non-linear theory in the quadratic approximation. It is shown that they required specific revision and correction by virtue of the fact that, when they are used in the solution of problems, spurious bifurcation points appear. In view of this, consistent geometrically non-linear equations of the theory of thin shells of the Timoshenko type are constructed in the quadratic approximation which enable one to investigate in a correct formulation both flexural as well as previously unknown non-classical forms of loss of stability (FLS) of thin plates and shells, many of which are encountered in practice, primarily in structures made of composite materials with a low shear stiffness. In the case of rectilinear elastic whereas, which are subjected to the action of conservative external forces and are made of an orthotropic material, the three-dimensional equations of the theory of elasticity are reduced to one-dimensional equations by using the Timoshenko model. Two versions of the latter equations are derived. The first of these corresponds to the use of the consistent version of the three-dimensional, geometrically non-linear relations in an incomplete quadratic approximation and the Timoshenko model without taking account of the transverse stretching deformations, and the second corresponds to the use of the three- dimensional relations in the complete quadratic approximation and the Timoshenko model taking account of the transverse stretching deformations. A series of new non-classical problems of the stability of columns is formulated and their analytical solutions are found using the equations which have been derived with the aim of analyzing their richness of content. Among these are problems concerning the torsional, flexural and shear FLS of a column in the case of a longitudinal axial, bilateral transverse and trilateral compression, a flexural-torsional FLS in the case of pure bending and axial compression together with pure bending and, also, a flexural FLS of a column in the case of torsion and a flexural-torsional FLS under conditions of pure shear. Five FLS of a cylindrical shell under torsion are investigated using the linearized neutral equilibrium equations which have been constructed: 1) a torsional FLS where the solution corresponding to it has a zero variability of the functions in the peripheral direction, 2) a purely beam bending FLS that is possible in the case of long shells and is accompanied by the formation of a single half-wave along the length of the shell while preserving the initial circular form of the cross-section, 3) a classical bending FLS, which is accompanied by the formation of a small number of half-waves in the axial direction and a large number of half-waves in a peripheral direction which is true in the case of long shells, 4) a classical bending FLS which holds in the case of short and medium length shells (the third and fourth types of FLS have already been thoroughly studied in the case of isotropic cylindrical shells), 5) a non-classical FLS characterized by the formation of a large number of shallow depressions in the axial as well as in the peripheral directions; the critical value of the torsional moment corresponding to this FLS is practically independent of the relative thickness of the shell. It is established that the well-known equations of the geometrically non-linear theory of shells, which were formulated for the case of the mean flexure of a shell, do not enable one to reveal the first, second and fifth non-classical FLS.

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The basic stages in the establishment of the non-linear theory of elasticity and, in particular, the non-linear theory of shells, plates and columns mainly date back to the first half of the last century and were marked in the Russian scientific literature by the coming to light of a series of remarkable monographs. Then, the fundamental equations of the geometrically non-linear theory of elasticity, both for arbitrary as well as for small deformations and displacements, were constructed by Novozhilov[1] and analyzed with the utmost clarity and completeness. In the case of small deformations but large displacements, the corresponding kinematic relations were later called the kinematic relations in the quadratic approximation. At the present time, these relations are presented and used without any doubts as being absolutely true and correct. According to these relations, the stretching and shear deformations are exactly equal to the difference in the components of the metric tensor prior to and after the deformation of a body if the Lagrangian method is used to describe them. The above mentioned difference in the components of the metric tensor, which are called the components of the strain tensor, is therefore adopted as the most convenient as a measure both finite and small deformations in all the current literature on the mechanics of a deformable solid and the correctness of this method of describing deformations has not given rise to any doubts.

The need for a revision of the kinematic relations in the theory of elasticity constructed in the quadratic approximation and, consequently, of all the non-linear equations in the mechanics of a deformable solid and, in particular, the two-dimensional equations of the theory of shells based on the definition of their finite and small deformations in the form of a difference in the components of the metric tensor prior to and after deformations of a body arose in connection with the appearance of spurious points of bifurcation in the solution of specific problems which had been formulated[2,4] using the above-mentioned kinematic relations constructed in the quadratic approximation. An analysis of these relations showed[3,4] that, of them, the relations serving to determine the stretching deformations E_α $(\alpha = 1, 2, 3)$[1] in the directions of the coordinate lines x^α, which are constructed with an accuracy $E_\alpha + 2 \approx 2$, are inconsistent and incorrect. A consistent version of the kinematic relations in the quadratic approximation was also constructed in Refs 3,4 and their use to construct consistent refined equations to describe the planar deformations of a strip-column and in the membrane theory of shells was subsequently described in Refs 5,6.

Analysis of the planar FLS of a column and the non-classical shear FLS of cylindrical shells, which were revealed on the basis of the refined equations, led to unexpected results, previously unknown in the theory of the stability of thin-walled structures, which enables us to formulate the following basic conclusions.

1. All the versions of the non-linear theory of shells, plates and columns for arbitrary displacements and rotations, developed up to the present time, which are based both on the classical Kirchhoff - Love and Bernoulli - Euler models as well as on refined models (in particular, Timoshenko models) in which the deformations are defined as the difference between the components of a metric tensor prior to and after deformation (see Refs 7–9, for example) are incorrect and inconsistent, since their use to solve specific problems leads to the appearance of spurious bifurcation points.
2. The well-known simplified versions of the theory of shells, plates and columns constructed by introducing some constraints or other imposed on the displacements and rotations (the non-linear theory of the mean flexure,[7–9] and the non-linear theory of shallow shells) are correct and consistent. However, they only essentially enable one to describe and investigate flexural FLS correctly.
3. The possibility of realizing some form of loss of stability of shells, plates and columns, of which certain non-classical FLS have been described previously[2,5,6] is completely determined by the form of the unperturbed stress-strain state, and the magnitudes of the critical loads depend very much on the values of the elastic and stiffness parameters of the material and the construction. Judging from the results obtained previously in Refs 5,6, the FLS for shells, plates and columns made of isotropic materials which have been revealed are not, as a rule, of practical interest. In this plan, the above-mentioned simplified versions of the non-linear theory of shells, based on the introduction of some or other constraints on the displacements and rotations, are to be considered as being finished up to a specified degree of completion only in the case of thin-walled structures made of isotropic materials.
4. Since the values of the critical loads, corresponding to some FLS or other from those revealed earlier in Refs 5,6, depend very much on the values of the shear moduli of the material then, in the light of all of the results which have been obtained,[3–6] there are, in the first place, pressing questions associated with the review, revision and development of new refined versions of the theory of columns, plates and shells made of composite materials with a low shear strength and also having a laminar structure.

It is well-known that, in the case of thin columns, plates and shells with a low shear strength, versions of the refined theory based on the Timoshenko model are simple and common. Up to the present time, many publications have dealt with their discussion for thin shells in some geometrically non-linear approximations and, in particular, in Refs 8–10. However, the above-mentioned drawbacks are characteristic of all of them. The results obtained below should be considered as the final result of a review of these versions.

1. Basic equations of the non-linear theory of thin, Timoshenko-type shells in the quadratic approximation

1.1. Kinematic relations

Introducing the triorthogonal system of curvilinear coordinates x^1, x^2, z, which is normally associated with the middle plane of a shell σ, we adopt the following notation: A_i, and k_i are the Lamé parameters in the plane σ and the principal curvatures, $H_i = A_i\,(1 + zk_i)$ are the Lamé parameters at an arbitrary point of the space of the shell at a distance z from σ, measured in the direction of the unit normal $\mathbf{m}$ to σ and $\mathbf{e}_i$ are the unit vectors in σ which, with the vector $\mathbf{m}$, constitute a right-handed basis ($i = 1, 2$).

If the vector of the displacements $\mathbf{U}$ of an arbitrary point $M(x^i, z)$ is represented by the expansion $\mathbf{U} = U_i\mathbf{e}_i + U_3\mathbf{m}$, then, in the case of small stretching strains ε_1, ε_2, ε_3 and arbitrary shear strains $\sin\gamma_{12}$, $\sin\gamma_{13}$, $\sin\gamma_{23}$, the following consistent kinematic relations hold in the quadratic approximations[4,5]:

$$\varepsilon_1 = E_{11} + (E_{12}^2 + E_{13}^2)/2, \quad \sin\gamma_{12} = E_{12}(1 + E_{22}) + E_{21}(1 + E_{11}) + E_{13}E_{23}; \quad \overrightarrow{\underleftarrow{1, 2, 3}} \tag{1.1}$$

In the case of a thin shell, when the approximate equalities $H_i \approx A_i$ occurring in relations (1.1) are taken with an accuracy $1 + zk_i \approx 1$, the quantities $E_{\alpha\beta}$ are determined using the formulae

$$\begin{aligned} &A_1E_{11} = U_{1,1} + A_{1,2}U_2/A_2 + A_1k_1U_3, \quad A_1E_{12} = U_{2,1} - A_{1,2}U_1/A_2; \quad \overrightarrow{\underleftarrow{1, 2}} \\ &E_{13} = U_{3,1}/A_1 - k_1U_1, \quad E_{31} = \partial U_1/\partial z; \quad \overrightarrow{\underleftarrow{1, 2}}; \quad E_{33} = \partial U_3/\partial z \end{aligned} \tag{1.2}$$

in which $(\ldots)_{,i} = \partial(\ldots)/\partial x^i$.

Henceforth, unless otherwise stated, the Latin indices i, j, k and s take the values of 1 and 2, and the Greek indices α and β take the values 1, 2 and 3.

With the aim of reducing the initial three-dimensional equations of the theory of elasticity to the two-dimensional equations of the theory of shells and plates, we will adopt the following approximations for the displacements occurring in relations (1.2)

$$U_i = u_i + z\gamma_i, \; U_3 = w + z\gamma \tag{1.3}$$

that correspond to the well-known refined Timoshenko model which, up to the present time, has been thoroughly studied within the framework of the initial three-dimensional kinematic relations of the theory of elasticity written for some degree of approximation.[8–12]

The following expressions for $E_{\alpha\beta}$ correspond to the approximations (1.3)

$$E_{ik} = e_{ik} + z\Omega_{ik}, \quad E_{i3} = \omega_i + z\Omega_{i3}, \quad E_{3i} = \gamma_i, \quad E_{33} = \gamma \tag{1.4}$$

in which

$$\begin{aligned} &A_1e_{11} = u_{1,1} + A_{1,2}u_2/A_2 + A_1k_1w, \quad A_1e_{12} = u_{2,1} - A_{1,2}u_1/A_2; \quad \overrightarrow{\underleftarrow{1, 2}} \\ &\omega_1 = w_{,1}/A_1 - k_1u_1, \quad \Omega_{13} = \gamma_{,1}/A_1 - k_1\gamma_1, \quad A_1\Omega_{12} = \gamma_{2,1} - A_{1,2}\gamma_1/A_2, \\ &A_1\Omega_{11} = \gamma_{1,1} + A_{1,2}\gamma_2/A_2 + A_1k_1\gamma; \quad \overrightarrow{\underleftarrow{1, 2}} \end{aligned} \tag{1.5}$$

If we restrict ourselves to the degree of accuracy in the approximation of the strains in the direction of the z axis which has previously been accepted (in Refs 8–12 and other papers), then, on substituting expressions (1.3) into relations (1.1), we arrive at the kinematic relations

$$\varepsilon_i = \varepsilon_{ii} + z\chi_{ii}, \quad \gamma_{12} \approx 2\varepsilon_{12} + 2z\chi_{12} \tag{1.6}$$

$$\varepsilon_3 = \varepsilon_{33} = \gamma + (\gamma_1^2 + \gamma_2^2)/2 \quad (1.7)$$

$$\gamma_{13} \approx 2\varepsilon_{13} = (1+\gamma)\omega_1 + (1+e_{11})\gamma_1 + e_{12}\gamma_2; \quad \overset{\longrightarrow}{\underset{\longleftarrow}{1,2}} \quad (1.8)$$

where

$$\begin{aligned} &\varepsilon_{11} = e_{11} + (e_{12}^2 + \omega_1^2)/2; \quad \overset{\longrightarrow}{\underset{\longleftarrow}{1,2}} \\ &2\varepsilon_{12} = 2\varepsilon_{21} = (1+e_{22})e_{12} + (1+e_{11})e_{21} + \omega_1\omega_2 \end{aligned} \quad (1.9)$$

$$\begin{aligned} &\chi_{11} = \Omega_{11} + e_{12}\Omega_{12} + \omega_1\Omega_{13}; \quad \overset{\longrightarrow}{\underset{\longleftarrow}{1,2}} \\ &2\chi_{12} + 2\chi_{21} = (1+e_{22})\Omega_{12} + (1+e_{11})\Omega_{21} + \omega_1\Omega_{23} + \omega_2\Omega_{13} + e_{12}\Omega_{22} + e_{21}\Omega_{11} \end{aligned} \quad (1.10)$$

In the scientific literature on the non-linear mechanics of shells, the relations

$$\varepsilon_3 = \gamma + (\gamma_1^2 + \gamma_2^2 + \gamma^2)/2 \quad (1.7a)$$

$$2\varepsilon_{ij} = e_{ij} + e_{ji} + e_{is}e_{js} + \omega_i\omega_j \quad (1.9a)$$

$$2\chi_{ij} = \Omega_{ij} + \Omega_{ji} + e_{is}\Omega_{js} + e_{js}\Omega_{is} + \omega_i\Omega_{j3} + \omega_j\Omega_{i3} \quad (1.10a)$$

are presented for the Timoshenko model instead of relations (1.7), (1.9) and (1.10), while the derived formulae (1.9) and (1.10) can be represented more compactly in the form

$$2\varepsilon_{ij} = e_{ij} + e_{ji} + e_{is}e_{js} + \omega_i\omega_j - \delta_{ij}e_j \quad (1.9b)$$

$$2\chi_{ij} = \Omega_{ij} + \Omega_{ji} + e_{is}\Omega_{js} + e_{js}\Omega_{is} + \omega_i\Omega_{j3} + \omega_j\Omega_{i3} - \delta_{ij}\Omega_j \quad (1.10b)$$

where (there is no summation over j) $e_j = e_{jj}e_{jj}$, $\Omega_j = e_{jj}\Omega_{jj}$.

Generally speaking, relations (1.7)–(1.10) hold for arbitrary displacements and finite shear strains but for small stretching strains, in all directions of the coordinate axes. The expressions for ε_{ii}, ε_3, χ_{ii} presented in them only differ slightly in form from the analogous expressions in (1.7a), (1.9a) and (1.10a) but are fundamentally different in content. This difference is associated with the fact that, in expressions (1.7), (1.9) and (1.10), there are no non-linear terms, which lead to the appearance of spurious bifurcation points in the solution of specific problems. This property of the derived relations is due to the correctness of the initial three-dimensional kinematic relations (1.1) on which they are based and some or other of the terms in them can only be neglected in special cases, which are determined by the form of the stress state of a shell, that occurs during the loading process, as can be shown starting from an analysis of earlier results.[5,6]

1.2. The equilibrium equations

We will assume that the boundary cuts of an open shell in the σ plane are bounded by the coordinate lines $x^i = x^i_-, x^i_+$ in which the vectors of the linear contour stresses and moments

$$\mathbf{Q}_i = Q_{i1}\mathbf{e}_1 + Q_{i2}\mathbf{e}_2 + Q_{i3}\mathbf{m}, \quad \mathbf{L}_i = L_{i1}\mathbf{e}_1 + L_{i2}\mathbf{e}_2 + L_{i3}\mathbf{m} \quad (1.11)$$

are specified and, at points of the surface σ, the vectors of the surface stresses and moments which have been reduced to it

$$\mathbf{X} = X_i\mathbf{e}_i + X_3\mathbf{m}, \quad \mathbf{M} = M_i\mathbf{e}_i + M_3\mathbf{m} \quad (1.12)$$

and refer to unit area of the surface σ, are specified. The variation of the work of these force and moments will be equal to

$$\delta A = \sum_{i=1}^{2}\int_{x_-^{3-i}}^{x_+^{3-i}} (\mathbf{Q}_i\delta\mathbf{u} + \mathbf{L}_i\delta\boldsymbol{\gamma})A_{3-i}dx^{3-i}\Big|_{x^i=x^i_-}^{x^i=x^i_+} + \iint_\sigma (\mathbf{X}\delta\mathbf{u} + \mathbf{M}\delta\boldsymbol{\gamma})A_1A_2dx^1dx^2 \quad (1.13)$$

where

$$\mathbf{u} = u_i\mathbf{e}_i + w\mathbf{m}, \quad \boldsymbol{\gamma} = \gamma_i\mathbf{e}_i + \gamma\mathbf{m} \tag{1.14}$$

If the linear internal stresses and moments ($2h$ is the thickness of the shell)

$$T_{\alpha\beta} = \int_{-h}^{h}\sigma_{\alpha\beta}dz, \quad M_{ik} = \int_{-h}^{h}\sigma_{ik}zdz \tag{1.15}$$

reduced to the surface σ, are introduced into the treatment, the formula

$$\delta U = \iint_{\sigma}(T_{ij}\delta\varepsilon_{ij} + 2T_{i3}\delta\varepsilon_{i3} + T_{33}\delta\varepsilon_3 + M_{ij}\delta\xi_{ij})A_1A_2dx^1dx^2 \tag{1.16}$$

holds for the variation of the strain potential energy of a thin shell. On introducing expressions (1.7), (1.8)–(1.10) here and the notation for the stresses

$$\begin{aligned} S_{11} &= T_{11} + T_{12}e_{21} + T_{13}\gamma_1 + M_{12}\Omega_{21}; \quad \overrightarrow{\underleftarrow{1,2}} \\ S_{12} &= T_{12}(1+e_{22}) + T_{11}e_{12} + T_{13}\gamma_2 + M_{11}\Omega_{12} + M_{12}\Omega_{22}; \quad \overrightarrow{\underleftarrow{1,2}} \\ S_{13} &= T_{13}(1+\gamma) + T_{11}\omega_1 + T_{12}\omega_2 + M_{11}\Omega_{13} + M_{12}\Omega_{23}; \quad \overrightarrow{\underleftarrow{1,2}} \\ N_{13} &= T_{13}(1+e_{11}) + T_{23}e_{21} + T_{33}\gamma_1; \quad \overrightarrow{\underleftarrow{1,2}}; \quad N_{33} = T_{33} + T_{13}\omega_1 + T_{23}\omega_2 \end{aligned} \tag{1.17}$$

and moments

$$H_{11} = M_{11} + M_{12}e_{21}, \; H_{12} = M_{21}(1+e_{22}) + M_{11}e_{12}, \; H_{13} = M_{11}\omega_1 + M_{12}\omega_2; \; \overrightarrow{\underleftarrow{1,2}} \text{ (1.18)} \tag{1.18}$$

we obtain

$$\delta U = \iint_{\sigma}(S_{ij}\delta e_{ij} + S_{i3}\delta\omega_i + H_{ij}\delta\Omega_{ij} + H_{i3}\delta\Omega_{i3} + N_{i3}\delta\gamma_i + N_{33}\delta\gamma)A_1A_2dx^1dx^2 \tag{1.19}$$

Using expressions (1.13) and (1.19) and taking account of relations (1.10) and (1.12), we now set up the variational equation of the principle of virtual displacements

$$\begin{aligned} \delta U - \delta A &= \sum_{i=1}^{2}\int_{x_-^{3-i}}^{x_+^{3-i}}[(S_{ij}-Q_{ij})\delta u_j + (S_{i3}-Q_{i3})\delta w + \\ &+ (H_{ij}-L_{ij})\delta\gamma_j + (H_{i3}-L_{i3})\delta\gamma]A_{3-i}dx^{3-i}\Big|_{x^i = x_-^i}^{x^i = x_+^i} - \\ &- \iint_{\sigma}(f_i\delta u_i + f_3\delta w + f_{3+i}\delta\gamma_i + f_6\delta\gamma)dx^1dx^2 = 0 \end{aligned} \tag{1.20}$$

from which the system of six differential equilibrium equations

$$\begin{aligned} f_1 &= (A_2S_{11})_{,1} + (A_1S_{21})_{,2} - A_{2,1}S_{22} + A_{1,2}S_{12} + A_1A_2(k_1S_{13} + X_1) = 0; \quad \overrightarrow{\underleftarrow{1,2}} \\ f_3 &= (A_2S_{13})_{,1} + (A_1S_{23})_{,2} - A_1A_2(k_1S_{11} + k_2S_{22} - X_3) = 0 \end{aligned} \tag{1.21}$$

$$\begin{aligned} f_{3+i} &= (A_2H_{11})_{,1} + (A_1H_{21})_{,2} - A_{2,1}H_{22} + A_{1,2}H_{12} + A_1A_2(k_1H_{13} - N_{13} + M_1) = 0; \quad \overrightarrow{\underleftarrow{1,2}} \\ f_6 &= (A_2H_{13})_{,1} + (A_1H_{23})_{,2} - A_1A_2(k_1H_{11} + k_2H_{22} + N_{33} - M_3) = 0 \end{aligned} \tag{1.22}$$

follows and the static boundary conditions on the shell edges $x_i' = x_-^i$, $x_i = x_+^i$

$$\begin{aligned} S_{ij} &= Q_{ij} \text{ When } \delta u_j \neq 0, \quad S_{i3} = Q_{i3} \text{ When } \delta w \neq 0 \\ H_{ij} &= L_{ij} \text{ When } \delta\gamma_j \neq 0, \quad H_{i3} = L_{i3} \text{ When } \delta\gamma \neq 0 \end{aligned} \tag{1.23}$$

Note that expressions (1.17) and (1.18) for the stresses and moments introduced into the treatment, which occur in Eqs (1.20)–(1.22) and the static boundary conditions (1.23), also differ from the corresponding expressions presented earlier (see for example, Refs 8–10). This difference, which is fundamental in relation to the richness of content and correctness of the equations constructed in the theory of Timoshenko-type shells, is entirely due to the fact that there is no $\gamma^2/2$ term in relation (1.7) and, in the first relations of (1.9) and (1.10), there are no terms $e_{ii}^2/2$ and $e_{ii}\Omega_{ii}$ respectively (there is no summation over i).

1.3. Elasticity relations

We shall assume that the shell material is linearly elastic and orthotropic and that the axes of orthotropy coincide with the axes of the chosen system of curvilinear coordinates. In the case of such a material, the components of the stresses are related to the strain components by the generalized Hooke's law relations[13]

$$\sigma_{\alpha 1} = g_{\alpha 1}\varepsilon_1 + g_{\alpha 2}\varepsilon_2 + g_{\alpha 3}\varepsilon_3, \quad \sigma_{\alpha\beta} = 2G_{\alpha\beta}\varepsilon_{\alpha\beta}\ (\alpha \neq \beta) \tag{1.24}$$

in which G_{12}, G_{13}, G_{23} are the shear moduli in the corresponding planes and $g_{\alpha\beta}$ are expressed in terms of the moduli of elasticity E_α and Poisson's ratios $\upsilon_{\alpha\beta}$ using the formulae

$$\begin{gathered} g_{11} = E_1(1-\upsilon_{23}\upsilon_{32})/\Delta,\ g_{12} = g_{21} = E_1(\upsilon_{21}+\upsilon_{23}\upsilon_{31})/\Delta = E_2(\upsilon_{12}+\upsilon_{13}\upsilon_{32})/\Delta; \\ \underleftarrow{\overrightarrow{1,2,3}} \\ \Delta = 1-\upsilon_{12}\upsilon_{21}-\upsilon_{23}\upsilon_{32}-\upsilon_{31}\upsilon_{13}-2\upsilon_{12}\upsilon_{23}\upsilon_{31} \end{gathered} \tag{1.25}$$

Substituting expressions (1.24) into equalities (1.15), we arrive at the two-dimensional elasticity relations for the stresses and moments

$$\begin{gathered} T_{\alpha\alpha} = C_{\alpha 1}\varepsilon_{11} + C_{\alpha 2}\varepsilon_{22} + C_{\alpha 3}\varepsilon_3, \quad T_{\alpha\beta} = 2B_{\alpha\beta}\varepsilon_{\alpha\beta}\ (\alpha\neq\beta) \\ M_{ii} = d_{i1}\chi_{11} + d_{i2}\chi_{22}, \quad M_{12} = 2D_{12}\chi_{12} \end{gathered} \tag{1.26}$$

in which the notation for the corresponding stiffnesses

$$C_{\alpha\beta} = 2hg_{\alpha\beta}, \quad B_{\alpha\beta} = 2hG_{\alpha\beta}, \quad d_{\alpha\beta} = 2h^3 g_{\alpha\beta}/3, \quad D_{12} = 2h^3 G_{12}/3 \tag{1.27}$$

has been introduced.

1.4. The basic equations of the theory of shells in the quadratic approximation, corresponding to the introduction of the assumption that $\sigma_{33}=0$

In the case of thin shells, use of the equality $\sigma_{33}=0$ corresponds to the assumption of the formation in them of a planar stress-strain state (SSS). In this case, relations (1.24) of the generalized Hooke's law can be represented in the form

$$\sigma_{11} = \tilde{E}_1(\varepsilon_1 + \upsilon_{21}\varepsilon_2); \quad \tilde{E}_1 = E_1/(1-\upsilon_{12}\upsilon_{21}), \quad \varepsilon_3 = -\tilde{\upsilon}_{13}\varepsilon_1 - \tilde{\upsilon}_{23}\varepsilon_2; \quad \underleftarrow{\overrightarrow{1,2}} \tag{1.28}$$

where

$$\tilde{\upsilon}_{13} = (\upsilon_{13}+\upsilon_{12}\upsilon_{23})/(1-\upsilon_{12}\upsilon_{21}); \quad \underleftarrow{\overrightarrow{1,2}}$$

Introducing expression (1.7) into equalities (1.24), we obtain the formula

$$\gamma = -\tilde{\upsilon}_{13}\varepsilon_1 - \tilde{\upsilon}_{23}\varepsilon_2 - (\gamma_1^2+\gamma_2^2)/2 \tag{1.29}$$

which, generally speaking, contradicts the initial approximation of the displacements and strains by expressions (1.3), (1.6) and (1.7) since the function γ is its left-hand side, which depends solely on x^1 and x^2, and the right-hand side is a function of all three coordinates x^1, x^2, z. On account of this, it is inadvisable to use the last formula of (1.28) and formulae (1.29) in the subsequent transformations of the basic equations which have been constructed in Sections

(1.1)–(1.3). Nevertheless, the simplified relations

$$\begin{aligned} &T_{11} = B_{11}(\varepsilon_{11} + \nu_{21}\varepsilon_{22}); \quad \underleftarrow{\overrightarrow{1,2}}; \quad T_{\alpha\beta} = 2B_{\alpha\beta}\varepsilon_{\alpha\beta}(\alpha \neq \beta), \quad T_{33} = 0 \\ &M_{11} = D_{11}(\chi_{11} + \upsilon_{21}\chi_{22}), \quad \underleftarrow{\overrightarrow{1,2}}, \quad M_{12} = 2D_{12}\chi_{12} \end{aligned} \tag{1.30}$$

where

$$B_{ii} = 2\tilde{E}_i h, \; D_{ij} = B_{ij}h^2/3, \; B_{11}\upsilon_{21} = B_{22}\upsilon_{12}, \; D_{11}\upsilon_{21} = D_{22}\upsilon_{12}, \; \tilde{E}_i = E_i(1 - \upsilon_{12}\upsilon_{21})^{-1}$$

are used within the framework of the approximation $\sigma_{33} = 0$, instead of the elasticity relations (1.26), in the mechanics of thin shells.

When account is taken of the equality $T_{33} = 0$ instead of the last relations of (1.17), we shall have

$$N_{13} = T_{13}(1 + e_{11}) + T_{23}e_{21}, \; N_{23} = T_{13}e_{12} + T_{23}(1 + e_{22}), \quad N_{33} = T_{13}\omega_1 + T_{23}\omega_2 \; (1.31) \tag{1.31}$$

while the first three relations of (1.17) and (1.18) remain unchanged.

2. Linearized neutral equilibrium equations of thin shells and their analysis

We will assume that, for a certain combination of external loads in an invariant direction, specified by the vectors (1.11) and (1.12), a solution of the equations constructed in Section 1 is found in the form of displacement functions $u_i^0, w_i^0, \gamma_i^0, \gamma^0$, stress functions $T_{\alpha\beta}^0$ and the moments M_{ij}^0. We linearize of the initial non-linear equations in the neighbourhood of this solution, retaining the previous notation for the increments in the functions which have been introduced into the treatment. As a result, assuming that the shell is stressed but undeformed up to its loss of stability, we obtain a system of linearized neutral equilibrium equations of the form (1.22) in which $X_\alpha = M_\alpha = 0$ and

$$\begin{aligned} &S_{11} = T_{11} + \underline{T_{12}^0 e_{21}} + T_{13}^0\gamma_1 + M_{12}^0\Omega_{21}; \quad \underleftarrow{\overrightarrow{1,2}} \\ &S_{12} = T_{12} + \underline{T_{12}^0 e_{22}} + \underline{T_{11}^0 e_{12}} + T_{13}^0\gamma_2 + M_{11}^0\Omega_{12} + M_{12}^0\Omega_{22}; \quad \underleftarrow{\overrightarrow{1,2}} \end{aligned} \tag{2.1}$$

$$S_{13} = T_{13} + T_{13}^0\gamma + T_{11}^0\omega_1 + T_{12}^0\omega_2 + M_{11}^0\Omega_{13} + M_{12}^0\Omega_{23}; \quad \underleftarrow{\overrightarrow{1,2}} \tag{2.2}$$

$$N_{13} = T_{13} + T_{13}^0 e_{11} + T_{23}^0 e_{21} + T_{33}^0\gamma_1; \quad \underleftarrow{\overrightarrow{1,2}}; \quad N_{33} = T_{33} + T_{13}^0\omega_1 + T_{23}^0\omega_2 \tag{2.3}$$

$$H_{11} = M_{11} + M_{12}^0 e_{21}; \; H_{12} = M_{12} + M_{12}^0 e_{22} + M_{11}^0 e_{12}, \; H_{13} = M_{11}^0\omega_1 + M_{12}^0\omega_2; \; \underleftarrow{\overrightarrow{1,2}} \tag{2.4}$$

As previously, the elasticity relations (1.30), in which

$$\varepsilon_i = e_{ii}, \quad 2\varepsilon_{12} = e_{12} + e_{21}, \quad 2\varepsilon_{i3} = \omega_i + \gamma_i, \quad \chi_{ii} = \Omega_{ii}, \quad 2\chi_{12} = \Omega_{12} + \Omega_{21} \tag{2.5}$$

hold for the stresses $T_{\alpha\beta}$ and the moments M_{ij} appearing in Eqs (2.1)–(2.4).

It can be seen from formulae (2.1)–(2.5) that, when they are used, the neutral equilibrium equations, expressed in terms of the required functions $u_i, w_i, \gamma_i, \gamma$, will have an extremely complex form. Their partial simplification is only possible with the introduction of some assumptions concerning the initial SSS and the possible forms of loss of stability of the shell. For example, in the case of the formation of only sheet subcritical stresses $T_{11}^0, T_{12}^0, T_{22}^0$ in a shell, formulae (2.1)–(2.4) can be represented in the simplified form

$$S_{ij} = T_{ij}, \quad S_{i3} = T_{i3} + T_{ij}^0\omega_j, \quad N_{i3} = T_{i3}, \quad H_{ij} = M_{ij}, \quad H_{i3} = 0 \tag{2.6}$$

if the underlined terms in relations (2.1) are neglected. If, moreover, the additional simplifying assumption $\sigma_{33} = 0$ is introduced in addition to (2.6) for the perturbed SSS in accordance with which $T_{33} = 0$ and, as in relations (1.30), taking account of the equalities (2.5)

$$T_{11} = B_{11}(e_{11} + \upsilon_{21}e_{22}), \quad \underleftarrow{\overrightarrow{1,2}}; \quad T_{12} = 2B_{12}\varepsilon_{12} = B_{12}(e_{12} + e_{21})$$

then, after eliminating the function γ, neutral equilibrium equations of the form

$$\begin{aligned} f_1 &= (A_2T_{11})_{,1} + (A_1T_{12})_{,2} - A_{2,1}T_{22} + A_{1,2}T_{12} + A_1A_2k_1S_{13} = 0; \quad \overrightarrow{\underleftarrow{1,2}} \\ f_3 &= (A_2S_{13})_{,1} + (A_1S_{23})_{,2} - A_1A_2(k_1T_{11} + k_2T_{22}) = 0 \\ f_{3+1} &= (A_2M_{11})_{,1} + (A_1M_{12})_{,2} - A_{2,1}M_{22} + A_{1,2}M_{12} - A_1A_2N_{13} = 0; \quad \overrightarrow{\underleftarrow{1,2}} \end{aligned} \tag{2.7}$$

are obtained in which

$$\begin{aligned} &M_{11} = D_{11}(\chi_{11} + \upsilon_{21}\chi_{22}); \quad \overrightarrow{\underleftarrow{1,2}}; \quad M_{12} = 2D_{12}\chi_{12} \\ &\chi_{11} = \frac{\gamma_{1,1}}{A_1} + \frac{A_{1,2}}{A_1A_2}\gamma_2; \quad \overrightarrow{\underleftarrow{1,2}}; \quad 2\chi_{12} = \frac{\gamma_{2,1}}{A_1} + \frac{\gamma_{1,2}}{A_2} - \frac{A_{1,2}}{A_1A_2}\gamma_1 - \frac{A_{2,1}}{A_1A_2}\gamma_2 \end{aligned} \tag{2.8}$$

Equations (2.7) only enable us to investigate flexural and the simplest shear FLS. As is well known, they correspond to the use of the initial geometrically non-linear equations of the theory of the mean flexure of a shell[8–10] when the use of the simplified relations

$$2\varepsilon_{i3} = \omega_i + \gamma_i, \quad 2\varepsilon_{ij} = e_{ij} + e_{ji} + \omega_i\omega_j, \quad 2\chi_{ij} = \Omega_{ij} + \Omega_{ji} \tag{2.9}$$

instead of the kinematic relations (1.8)–(1.12) is admissible. However, if only stresses T^0_{ij} are formed in a shell which corresponds to its initial zero-moment state, discarding of the underlined terms in relations (2.1) is equally inadmissible which, as shown by an analysis of the results obtained earlier,[5,6] leads to a loss of the richness of content of the neutral equilibrium equations, which have been constructed as regards the possibility of describing and revealing a number of other forms of loss of stability on the basis of these equations. The dropping of other parametric terms of the form $M^0_{11}\Omega_{13}, \ldots, T^0_{i3}e_{ij}, \ldots$, etc. in relations (2.1)–(2.4) when the parameters of the initial SSS of the form M^0_{ij}, T^0_{i3}, T^0_{33} are formed in shells or plates is also inadmissible in the general case, as follows from an analysis of the previous results.[5,6]

The solutions of a number of new non-classical problems of the stability of shells and columns presented below serve as a confirmation of the conclusion which has been formulated.

3. The stability of a circular ring in the case of pure shear

Consider a ring, which has a radius of the middle surface R, referred to by an angular coordinate $x^2 = \theta$. We shall assume that it finds itself due to conditions of pure shear due to the action of surface forces q applied to the faces $z = \pm h$ and directed in opposite directions. We will assume that the equalities $\sigma^0_{11} = \sigma_{11} = 0$ hold in the axial direction $x^1 = x$. Elastic relations of the perturbed state of the following form

$$T_{22} = B_{22}(\varepsilon_{22} + \upsilon_{32}\varepsilon_3), \; T_{33} = B_{33}(\varepsilon_3 + \upsilon_{23}\varepsilon_{22}), \; T_{23} = 2B_{23}\varepsilon_{23}, \quad M_{22} = D_{22}\chi_{22} \tag{3.1}$$

in which

$$B_{22} = 2\tilde{E}_2h, \quad B_{33} = 2\tilde{E}_3h, \quad D_{22} = B_{22}h^2/3, \quad B_{23} = 2hG_{23} \tag{3.2}$$

hold for such a planar SSS and, on introducing the notation $(\ldots)' = d(\ldots)/d\theta$, $u_2 = \upsilon$, $\gamma_2 = \psi$, we obtain formulae for determining the components of the linear axial shear and flexural strains

$$\varepsilon_{22} = e_{22} = \frac{\upsilon' + w}{R}, \quad \varepsilon_2 = \gamma, \quad \chi_{22} = \frac{\psi' + \gamma}{R}, \quad \omega_2 = \frac{w' + \upsilon}{R}, \quad 2\varepsilon_{23} = \frac{w' - \upsilon}{R} + \psi \tag{3.3}$$

In the case being considered, the external surface stresses and moments, occurring in the subcritical equilibrium equation, have the form

$$Z_1 = X_2 = X_3 = 0, \quad M_1 = 0, \quad M_2 = 2hq$$

and the internal stresses and moments of the initial unperturbed stress state in the initial approximation are determined using the formulae

$$T^0_{11} = T^0_{13} = T^0_{33} = T^0_{22} = M^0_{22} = T^0_{33} = 0, \quad T^0_{23} = M_2 = 2hq \tag{3.4}$$

By virtue of the formulae for the stresses and moments characterizing the perturbed SSS of the ring, we obtain

$$\begin{aligned}&S_{22} = T_{22} + \underline{T_{23}^0\psi}, \quad S_{23} = T_{23} + \underline{T_{23}^0\gamma}, \quad N_{23} = T_{23} + \underline{T_{23}^0 e_{22}}, \quad N_{33} = T_{33} + \underline{T_{23}^0\omega_2},\\ &H_{22} = M_{22}, \quad H_{23} = 0\end{aligned} \tag{3.5}$$

When formulae (3.5), which were constructed in Section 2, are used, the neutral equilibrium equations take the form

$$\begin{aligned}&f_2 = (T_{22} + T_{23}^0\psi)' + T_{23} + T_{23}^0\gamma = 0, \quad f_3 = (T_{23} + T_{23}^0\gamma)' - T_{22} - T_{23}^0\psi = 0\\ &f_5 = M_{22}' - R(T_{23} + T_{23}^0 e_{22}) = 0, \quad f_6 = M_{22} + R(T_{33} + T_{23}^0\omega_2) = 0\end{aligned} \tag{3.6}$$

in which, in accordance with relations (3.1) and (3.3),

$$\begin{aligned}&T_{22} = B_{22}\left(\frac{\upsilon' + w}{R} + \upsilon_{32}\gamma\right), \quad T_{33} = B_{33}\left(\gamma + \upsilon_{23}\frac{\upsilon' + w}{R}\right), \quad T_{23} = B_{23}\left(\frac{\upsilon' - w}{R} + \gamma_2\right),\\ &M_{22} = D_{22}\frac{\psi' + \gamma}{R}\end{aligned} \tag{3.7}$$

3.1. The shear FLS of a ring with zero variability of the parameters of the perturbed SSS in a peripheral direction

The equations describing the FLS being considered follow from relations (3.3), (3.6) and (3.7) if we formally put $d(\ldots)/d\theta = (\ldots)' = 0$ in them. This is equivalent to the representation of the solutions of the above mentioned equations in the form

$$\upsilon = \tilde{\upsilon}_n\cos n\theta, \quad w = w_n\sin n\theta, \quad \psi = \tilde{\psi}_n\cos n\theta, \quad \gamma = \gamma_n\sin n\theta \tag{3.8}$$

from which, by analogy with what has been reported earlier,[14] the solution when $n = 0$ corresponds to a shear FLS and the solution when $n = 2, 3, \ldots$ to a flexural FLS.

When $(\ldots)' = 0$, from Eqs (3.6) taking account of equalities (3.7) we obtain the equations

$$T_{23} + T_{23}^0\gamma = B_{23}\left(\gamma_2 - \frac{w}{R}\right) + T_{23}^0\gamma = 0, \quad T_{22} + T_{23}^0\psi = B_{22}\left(\frac{w}{R} + \upsilon_{32}\gamma\right) + T_{23}^0\psi = 0$$

$$T_{23} + T_{23}^0 e_{22} = B_{23}\left(\gamma_2 - \frac{w}{R}\right) + T_{23}^0 e_{22} = 0$$

$$\frac{D_{22}}{R^2}\gamma + T_{33} + T_{23}^0\omega_2 = \frac{D_{22}}{R^2}\gamma + B_{33}\left(\gamma + \upsilon_{23}\frac{w}{R}\right) + T_{23}^0\omega_2 = 0$$

from which the bifurcation value

$$\overset{s}{q}_* = \sigma_{23}^{*s} = \frac{B_{23}}{2h}\sqrt{\frac{B_{33}(1 + \upsilon_{23}) + B_{22}(1 + \upsilon_{32})}{B_{23}}} = G_{23}\sqrt{\frac{E_3(1 + \upsilon_{23}) + E_2(1 + \upsilon_{32})}{G_{23}(1 - \upsilon_{23}\upsilon_{32})}} \tag{3.9}$$

corresponding to the shear FLS of the ring follows with an accuracy $B_{33} + B_{22}h^2/(3R^2) \approx B_{33}$. Since $G_{23} < E_2$, $G_{23} < E_3$, it follows from this that $\sigma_{23}^{*s} > G_{23}$.

It should be mentioned that, when the initial simplifying assumption $\gamma = 0$ is introduced, which is often used in the literature when constructing refined models in the theory of shells, we only arrive at the trivial solution $\upsilon = w = \gamma_2 = 0$. This result confirms the conclusion that making any simplifications to the non-linear equations of the theory of shells and to the linearized equations of stability theory requires very great care and there must be serious grounds for doing so. On the other hand, the solution $\upsilon = w = \psi = 0$, which results when $\gamma = 0$, means that it is impossible to realize the shear FLS being considered in a ring joined on its faces to bodies which are absolutely rigid in a radial direction.

3.2. The flexural FLS of a ring when $n \neq 0$

The equations $f_2 = 0$, $f_3 = 0$ of system (3.6) can be represented in the form

$$(T_{22} + T_{23}^0\psi)'' + T_{22} + T_{23}^0\gamma_2 = 0, \quad (T_{23} + T_{23}^0\gamma)'' + T_{23} + T_{23}^0\gamma = 0 \tag{3.10}$$

which, using the elasticity relations, we shall write in the form

$$
\begin{aligned}
&B_{22}(e_{22}+\upsilon_{32}\gamma)''+B_{22}(e_{22}+\upsilon_{32}\gamma)+T_{23}^0\psi''+T_{23}^0\psi = 0\\
&B_{23}(\psi+\omega_2)''+B_{23}(\psi+\omega_2)+T_{23}^0\gamma''+T_{23}^0\gamma = 0
\end{aligned}
\tag{3.11}
$$

By analogy with these equations, the last two equations of system (3.6), which are solved for the functions e_{22}, γ, ω_2, ψ, take the form

$$
D_{22}\frac{\psi''+\gamma'}{R}-RB_{23}(\psi+\omega_2)-RT_{23}^0e_{22} = 0,\ D_{22}\frac{\psi'+\gamma}{R}+RB_{33}(\psi+\upsilon_{23}e_{22})+RT_{23}^0\omega_2 = 0
\tag{3.12}
$$

The structure of Eq. (3.12) indicates that it is more convenient to represent the solutions of the problem being considered in the form

$$
e_{22} = e_n\sin n\theta+\tilde{e}_n\cos n\theta,\quad \omega_2 = \omega_n\sin n\theta+\tilde{\omega}_n\cos n\theta
$$

$$
\psi = \psi_n\sin n\theta+\tilde{\psi}_n\cos n\theta,\quad \gamma = \gamma_n\sin n\theta+\tilde{\gamma}_n\cos n\theta
$$

When these representations from equations (3.11) are used, we obtain the equalities

$$
\begin{aligned}
&\psi_n+\omega_n = -\frac{T_{23}^0}{B_{23}}\gamma_n,\quad \tilde{\psi}_n+\tilde{\omega}_n = -\frac{T_{23}^0}{B_{23}}\tilde{\gamma}_n,\quad \omega_n = -\psi_n-\frac{T_{23}^0}{B_{23}}\gamma_n\\
&\tilde{\omega}_n = -\tilde{\psi}_n-\frac{T_{23}^0}{B_{23}}\tilde{\gamma}_n,\quad e_n = -\upsilon_{32}\gamma_n-\frac{T_{23}^0}{B_{22}}\psi_n,\quad \tilde{e}_n = -\upsilon_{32}\bar{\tilde{\gamma}}n-\frac{T_{23}^0}{B_{22}}\tilde{\psi}_n
\end{aligned}
\tag{3.13}
$$

Similarly, the differential equations (3.12) are reduced to algebraic equations and, after using relations (3.13) and carrying out some reduction, take the form

$$
\begin{aligned}
&a\psi_n+bn\tilde{\gamma}_n-c\gamma_n = 0,\quad a\tilde{\psi}_n-bn\gamma_n-c\tilde{\gamma}_n = 0\\
&d\gamma_n-e\psi_n-bn\tilde{\psi}_n = 0,\quad d\tilde{\gamma}_n-e\tilde{\psi}_n+bn\psi_n = 0
\end{aligned}
\tag{3.14}
$$

Here,

$$
b = \frac{D_{22}}{R^2},\quad a = \frac{D_{22}}{R^2}n^2-\frac{(T_{23}^0)^2}{B_{22}} = bn^2-\frac{(T_{23}^0)^2}{B_{22}},\quad c = T_{23}^0(1+\upsilon_{32})
$$

$$
d = \frac{D_{22}}{R^2}+\tilde{B}_{33}\frac{(T_{23}^0)^2}{B_{23}} = b+\tilde{B}_{33}-\frac{(T_{23}^0)^2}{B_{23}},\quad e = T_{23}^0\left(1+\frac{B_{23}\upsilon_{23}}{B_{22}}\right)
$$

$$
\tilde{B}_{33} = \tilde{B}_{33}(1-\upsilon_{23}\upsilon_{32}) = 2E_3h
$$

The characteristic equation

$$
c^2-\left(\frac{ade}{b^2n^2-e^2}\right)^2+b^2n^2\left(\frac{ad}{b^2n^2-e^2}-1\right)^2 = 0
\tag{3.15}
$$

gives the condition for the solution of system (3.14) to be non-trivial. Formula (3.9), which corresponds to a purely shear FLS of the ring, follows from this characteristic equation when $n=0$. When $n\neq 0$, the determination of the quantity $2hq_*^f$, which corresponds to a flexural FLS, from Eq. (3.15) in the form of a compact formula is exceedingly complicated. Taking account of the structure of equalities (3.13), we therefore represent the first integrals of Eq. (3.10) in the form

$$
T_{22} = -T_{23}^0\psi,\quad T_{23} = -T_{23}^0\gamma
$$

and the formulae

$$
\psi+\omega_2 = -\frac{T_{23}^0\gamma}{B_{23}},\quad \omega_2 = -\psi-\frac{T_{23}^0\gamma}{B_{23}},\quad e_{22} = -\upsilon_{32}\gamma-\frac{T_{23}^0\psi}{B_{22}}
\tag{3.16}
$$

follow from this when the elasticity relations are used.

Now introducing expressions (3.16) into equalities (3.12), we arrive at the system of two differential equations

$$\begin{aligned} &\frac{1}{R^2}D_{22}(\psi''+\underline{\gamma'})+T_{23}^0(1+\upsilon_{32})\gamma+\frac{(T_{23}^0)^2}{B_{22}}\gamma_2 = 0 \\ &\frac{1}{R}D_{22}\underline{\psi'}+\left(\tilde{B}_{33}-\frac{(T_{23}^0)^2}{B_{23}}\right)\gamma-\left(1+\frac{B_{33}\upsilon_{23}}{B_{22}}\right)T_{23}^0\gamma_2 = 0 \end{aligned} \tag{3.17}$$

written with an accuracy $B_{33}+B_{22}h^2/(3R^2)\equiv(\ldots)'=B_{33}$.

We shall seek its solution in the form

$$\psi = \Psi_n\exp(-in\theta),\quad \gamma = \Gamma_n\exp(-in\theta) \tag{3.18}$$

which corresponds to neglecting the underlined terms in system (3.17), which has practically no effect on the accuracy of the results obtained in the theory of thin shells. We now introduce expressions (3.18) into system (3.17) and equate the real parts of the resulting algebraic equations to zero, which leads to a system for which the characteristic equation gives the condition for the solution to be non-trivial. If the subcritical stress T_{23}^0 occurring in this equation is expressed in terms of the load parameter m in accordance with the formula

$$T_{23}^0 = m\sqrt{B_{23}[B_{33}(1+\upsilon_{23})+B_{22}(1+\upsilon_{32})]}$$

then the characteristic equation in m

$$m^4-(1+r_{22})m^2+r_{33} = 0 \tag{3.19}$$

is obtained, where

$$r_{22} = g_2\chi\varepsilon^2n^2,\quad r_{33} = g_3(1-\upsilon_{23}\upsilon_{32})\chi^2\varepsilon^2n^2$$

$$\varepsilon^2 = \frac{h^2}{3R^2},\quad g_2 = \frac{B_{22}}{B_{23}},\quad g_3 = \frac{\tilde{B}_{33}}{B_{23}} = \frac{E_3}{G_{23}},\quad \chi = \frac{\upsilon_{23}}{\upsilon_{32}+\upsilon_{23}+2\upsilon_{23}\upsilon_{32}}$$

Since $B_{22}\approx B_{33}\approx\tilde{B}_{33}$, $\upsilon_{23}<1$, $\varepsilon\ll 1$, the smallest positive value of the root m_* of Eq. (3.19) is determined using the formula

$$m_* = \left(\frac{1+r_{22}}{2}-\sqrt{\left(\frac{1+r_{22}}{2}\right)^2-r_{33}}\right)^{1/2}$$

and corresponds to the loss of stability of the ring when $n\neq 0$.

The results of calculations of the values of m_* for different values of g_2, g_3, υ_{23} and ε, which characterize the degree of orthotropy of the material and the relative thickness of the ring, are shown in Table 1. Since, according to the representation which has been adopted $q=T_{23}^0/(2h)=mq_*^s$, then, as follows from Table 1, the critical values of the load q_*^f, corresponding to a flexural FLS, are significantly lower than the critical values q_*^s corresponding to a shear FLS of the ring.

If the underlined terms in relations (3.5) are neglected, which is equivalent to the construction of the equations for the neutral equilibrium of the ring starting out from the geometrically non-linear kinematic relations in their simplest form, as has been adopted and is very widely used in the literature, then all the parametric terms in the resulting

Table 1

g_2	10				200			
g_3	3		50		3		50	
υ_{23}	0.2	0.4	0.2	0.4	0.2	0.4	0.2	0.4
$\varepsilon=0.01$								
$m_*^f\times 10^3$	12.12	10.97	7.90	3.16	16.79	16.98	0.518	0.475
$\varepsilon=0.1$								
$m_*^f\times 10^2$	11.80	10.69	7.88	3.15	9.88	9.83	33.87	31.65

resolvents disappear. Consequently, the formulation of the problem of the stability of a ring under conditions of pure shear on the basis of the geometrically non-linear equations of the theory of flexure of shells is, in general, impossible.

4. Equations and problems of the stability of columns, based on the use of kinematic relations in an incomplete quadratic approximation

There are very extensive investigations associated with the construction of the geometrically non-linear equations of the theory of elastic and inelastic columns in the case of arbitrary displacements and, also, with the study of their FLS under the action of a different form of conservative and non-conservative loads. The fundamental results in this domain are associated with the names of Euler, Nikolai, Dzhanedlidze, Feodos'ev, Bolotin, Pfluger and many others. For information, we merely draw attention to the monograph of Bolotin[15] in which certain historical information and the results of the solution of a number of problems concerning the stability of columns under non-standard forms of loading are presented. In particular, the problem of the stability of the planar form of bending of columns, which is even presented in certain textbooks for a course on the strength of materials, and the problem of the stability of the equilibrium of a twisted column can be included in such problems. It would appear that everything is extremely clear and has been known for a long time in the field of the theory of stability of columns. However, the results obtained below indicate that this is not so.

4.1. Neutral equilibrium equations

In rectangular Cartesian coordinates x^1, x^2, x^3, kinematic relations in an incomplete quadratic approximation of the following form[2,3]

$$\varepsilon_1 = E_{11} + (E_{12}^2 + E_{13}^2)/2\ldots, \quad \gamma_{12} = E_{12} + E_{21} + E_{13}E_{23}, \ldots; \quad E_{\alpha\beta} = \partial u_\alpha/\partial x^\beta \tag{4.1}$$

hold for determining the stretching deformations of a column ε_1, ε_2, ε_3 and the shear strains γ_{12}, γ_{13}, γ_{23}.

When they are used, the relation[2,3] between the stress components $\sigma_{\alpha\beta}$ in the deformed axes and the stress components $\sigma_{\alpha\beta}^0$ in the undeformed axes x^α

$$\begin{aligned}&\sigma_{11}^* = \sigma_{11} + \sigma_{12}E_{21} + \sigma_{13}E_{31}, \quad \sigma_{12}^* = \sigma_{12} + \sigma_{11}E_{12} + \sigma_{13}E_{32},\\ &\sigma_{13}^* = \sigma_{13} + \sigma_{11}E_{13} + \sigma_{12}E_{23}; \quad \overrightarrow{\underleftarrow{1, 2, 3}}\end{aligned} \tag{4.2}$$

are established, where $\sigma_{\alpha\beta}^* \neq \sigma_{\beta\alpha}^*$ while $\sigma_{\alpha\beta} = \sigma_{\beta\alpha}$.

If the equations based on the use of relations (4.1) and (4.2) are linearized in the neighbourhood of a certain initial stress state with stress components $\sigma_{\alpha\beta}^*$, formed under the action of conservative external forces, then the variational equation

$$\iiint_V \sigma_{\alpha\beta}^* \delta E_{\alpha\beta} dV = 0 \tag{4.3}$$

holds in the neutral equilibrium state without taking account of the parametric strain terms. In Eq. (4.3), the previous notation has been retained for small increments in the displacements U_α, the quantities $E_{\alpha\beta} = \partial U_\alpha/\partial x^\beta$ and the stress components $\sigma_{\alpha\beta}^*$ but, unlike relations (4.2) and (1.24),

$$\begin{aligned}&\sigma_{11}^* = \sigma_{11} + \sigma_{12}^0E_{21} + \sigma_{13}^0E_{31}, \quad \sigma_{12}^* = \sigma_{12} + \sigma_{11}^0E_{12} + \sigma_{13}^0E_{32},\\ &\sigma_{13}^* = \sigma_{13} + \sigma_{11}^0E_{13} + \sigma_{12}^0E_{23}; \quad \overrightarrow{\underleftarrow{1, 2, 3}}\end{aligned} \tag{4.4}$$

$$\sigma_{\alpha\alpha} = g_{\alpha1}E_{11} + g_{\alpha2}E_{22} + g_{\alpha3}E_{33}, \quad \sigma_{12} = G_{12}(E_{12} + E_{21}), \ldots \tag{4.5}$$

Next, introducing the notation $x^1 = x$, $\alpha^2 = y$, $x^3 = z$ and assuming that, in each cross-section of the column $x = \text{const}$, the y and z axes are principal central axes of inertia, we adopt the following representation for the displacement vector

$\mathbf{U} = U_1\mathbf{i} + U_2\mathbf{j} + U_3\mathbf{k}$

$$\begin{aligned} \mathbf{u} &= u(x)\mathbf{i} + \upsilon(x)\mathbf{j} + w(x)\mathbf{k} + \boldsymbol{\varphi}(x)\times\boldsymbol{\rho} = u\mathbf{i} + \upsilon\mathbf{j} + w\mathbf{k} + \\ &+ (\varphi\mathbf{i} + \psi\mathbf{j} + \chi\mathbf{k})\times(y\mathbf{j} + z\mathbf{k}) = (u + z\psi - y\chi)\mathbf{i} + (\upsilon - z\varphi)\mathbf{j} + (w + y\varphi)\mathbf{k} \end{aligned} \tag{4.6}$$

and, using this, we have

$$\begin{aligned} &E_{11} = u' + z\psi' - y\chi', \quad E_{12} = \upsilon' - z\varphi', \quad E_{13} = w' + y\varphi' \\ &E_{21} = -\chi, \quad E_{23} = \varphi, \quad E_{31} = \psi, \quad E_{32} = -\varphi, \quad E_{22} = E_{33} = 0 \end{aligned} \tag{4.7}$$

It should be noted that the representation adopted corresponds to the well-known Timoshenko kinematic model in its simplest form which, when used in accordance with formulae (4.7), yields zero strain increments

$$\gamma_{23} = E_{23} + E_{32} = 0, \quad \varepsilon_2 = E_{22} = 0, \quad \varepsilon_3 = E_{33} = 0$$

(the kinematic hypotheses of the theory of columns). In addition to this, we adopt the hypotheses

$$\sigma_{22} = 0, \quad \sigma_{33} = 0$$

which are classical in the theory of columns and, when used, lead to the Hooke's law relation

$$\sigma_{11} = E_1\varepsilon_1 = E_1E_{11}$$

As a result, assuming that $\sigma^0_{23} = 0$ in the initial state and taking account of relations (4.7), instead of relations (4.4) we obtain the reduced relations

$$\begin{aligned} &\sigma^*_{11} = E_1(u' + z\psi' - y\chi') - \sigma^0_{12}\chi + \sigma^0_{13}\psi \\ &\sigma^*_{12} = (G_{12} + \sigma^0_{11})(\upsilon' - z\varphi') - G_{12}\chi - \sigma^0_{13}\varphi, \ \ldots, \quad \sigma^*_{32} = -\sigma^0_{33}\varphi + \sigma^0_{13}(\upsilon' - z\varphi') \end{aligned} \tag{4.8}$$

and, on substituting expressions (4.7) into Eq. (4.3), we arrive at a variational equation of the form

$$\int_0^a (Q^*_x\delta u' + M^*_y\delta\psi' + M^*_z\delta\chi' + Q^*_y\delta\upsilon' + Q^*_z\delta w' + M^*_x\delta\varphi' + N^*_z\delta\psi + N^*_y\delta\chi + N^*_x\delta\varphi)dx = 0 \tag{4.9}$$

where the following notation for the internal stresses and moments is introduced (henceforth, unless otherwise stated, integration is carried out over the cross-section area of the bar F)

$$\begin{aligned} &Q^*_x = \iint\sigma^*_{11}dF, \quad Q^*_y = \iint\sigma^*_{12}dF, \quad Q^*_z = \iint\sigma^*_{13}dF \\ &M^*_y = \iint\sigma^*_{11}z\,dF, \quad M^*_z = -\iint\sigma^*_{11}y\,dF, \quad M^*_x = \iint(\sigma^*_{13}y - \sigma^*_{12}z)dF \\ &N^*_z = \iint\sigma^*_{31}dF, \quad N^*_y = -\iint\sigma^*_{21}dF, \quad N^*_x = \iint(\sigma^*_{23} - \sigma^*_{32})dF \end{aligned} \tag{4.10}$$

After standard reduction, the homogeneous neutral equilibrium equations of the column in stresses and moments

$$\frac{dQ^*_x}{dx} = 0, \ \frac{dQ^*_y}{dx} = 0, \ \frac{dQ^*_z}{dx} = 0, \ \frac{dM^*_y}{dx} - N^*_z = 0, \ \frac{dM^*_z}{dx} - N^*_y = 0, \ \frac{dM^*_x}{dx} - N^*_x = 0 \tag{4.11}$$

follow from the equality (4.9) and the boundary conditions in the sections $x = 0$, $x = a$

$$\begin{aligned} &Q^*_x = 0 \text{ When } \delta u \neq 0, \quad Q^*_y = 0 \text{ When } \delta\upsilon \neq 0, \quad Q^*_z = 0 \text{ When } \delta w \neq 0 \\ &M^*_y = 0 \text{ When } \delta\psi \neq 0, \quad M^*_z = 0 \text{ When } \delta\chi \neq 0, \quad M^*_x = 0 \text{ When } \delta\varphi \neq 0 \end{aligned} \tag{4.12}$$

We will assume that the initial stresses σ^0_{12}, σ^0_{13}, σ^0_{22}, σ^0_{33} appearing in relations (4.8) are specified by values which have been averaged over a cross-section and that the initial normal stress σ^0_{11} in the first approximation is determined by the strength of materials formula

$$\sigma^0_{11} = Q^0_x/F + yM^0_zJ_z + zM^0_y/J_y \tag{4.13}$$

in which Q^0_x, M^0_z, M^0_y are the axial force and the internal bending moments in the cross-sections $x = \text{const}$, and J_y, J_z are the moments of inertia of the cross-section with respect to the principal central axes.

If it is assumed that the configuration of the cross-sections of a column satisfy the conditions

$$\iint y^3 dF = 0, \quad \iint z^3 dF = 0$$

then, by virtue of the assumptions made, when expressions (4.8) are substituted into formulae (4.10), we arrive at elastic relations of the following form

$$\begin{aligned}
&Q_x^* = F(E_1 u' - \sigma_{12}^0 \chi + \sigma_{13}^0 \psi), \quad Q_y^* = G_{12} F(\upsilon' - \chi) + Q_x^0 \upsilon' - M_y^0 \varphi' - \sigma_{13}^0 F \varphi \\
&Q_z^* = G_{13} F(w' + \psi) + Q_x^0 w' + M_z^0 \varphi' + \sigma_{12}^0 F \varphi \\
&M_y^* = E_1 J_y \psi', \quad M_z^* = E_1 J_z \chi' \\
&M_x^* = [(G_{13} + Q_x^0/F) J_z + (G_{12} + Q_x^0/F) J_y] \varphi' + M_z^0 w' - M_y^0 \upsilon' \\
&N_z^* = F[(G_{13} + \sigma_{33}^0)\psi + G_{13} w'], \quad N_y^* = F[(G_{12} + \sigma_{22}^0)\chi - G_{12}\upsilon'] \\
&N_x^* = F[(\sigma_{22}^0 + \sigma_{33}^0)\varphi + \sigma_{12}^0 w' - \sigma_{13}^0 \upsilon']
\end{aligned} \tag{4.14}$$

It is clear that, in the general case, when

$$M_z^0 \neq 0, \quad M_y^0 \neq 0, \quad Q_x^0 \neq 0, \quad \sigma_{12}^0 \neq 0, \quad \sigma_{13}^0 \neq 0, \quad \sigma_{22}^0 \neq 0, \quad \sigma_{33}^0 \neq 0$$

on substituting expressions (4.14) into Eq. (4.11) a connected system of homogeneous differential equations in the six required functions u, υ, w, φ, ψ, χ is obtained. Separation of the system is only possible in special cases of the loading of the column, the treatment of which is of the greatest practical interest.

In particular, when $\sigma_{12}^0 = 0$, $\sigma_{13}^0 = 0$, the first equation of (4.11), which takes the form $(E_1 F u)' = 0$, will only have a trivial solution and, when

$$E_1 F = \text{const}, \quad D_2 = \text{const}, \quad D_1 = \text{const}, \quad Q_x^0 = \text{const}, \quad M_y^0 = \text{const}, \quad M_z^0 = \text{const}$$

the remaining equations can be represented in the form

$$D_2 \psi'' - G_{13}(w' + \psi) - \sigma_{33}^0 \psi = 0, \quad G_{13} F(w'' + \psi') + Q_x^0 w'' + M_z^0 \varphi'' = 0 \tag{4.15}$$

$$D_1 \chi'' - G_{12}(\chi - \upsilon') - \sigma_{22}^0 \chi = 0, \quad G_{12} F(\upsilon'' - \chi') + Q_x^0 \upsilon'' - M_y^0 \varphi'' = 0 \tag{4.16}$$

$$(B_p + J_p Q_x^0/F)\varphi'' + M_z^0 w'' - M_y^0 \upsilon'' - (\sigma_{22}^0 + \sigma_{33}^0) F \varphi = 0 \tag{4.17}$$

where

$$D_2 = E_1 J_y/F, \quad D_1 = E_1 J_z/F, \quad B_p = G_{13} J_z + G_{12} J_y, \quad J_p = J_y + J_z \tag{4.18}$$

It is seen that the torsional rigidity of a column made of an orthotropic material is characterized by the quantity B_p, and J_p is the polar moment of inertia of a cross-section.

We will now consider the solutions of Eqs (4.15)–(4.17) for a number of special forms of loading of the column.

4.2. *Uniform axial compression of a column by a stress $\sigma_{11}^0 = -p$*

Since $Q_x^0 = -pF$ in the given case, Eqs (4.15)–(4.17) take the form

$$D_2 \psi'' - G_{13}(w' + \psi) = 0, \quad G_{13}(w'' + \psi') - p w'' = 0 \tag{4.19}$$

$$D_1 \chi'' - G_{12}(\chi - \upsilon') = 0, \quad G_{12}(\upsilon'' - \chi') - p \upsilon'' = 0 \tag{4.20}$$

$$(B_p - p J_p)\varphi'' = 0 \tag{4.21}$$

Solutions of equations of the type (4.19) and (4.20) are well known;[16] in the case of a column with hinged edges they yield four bifurcation values of the stress p

$$p_*^{(i)} = D_i \pi^2 (a^2 + D_i \pi^2 / G_{1(1+i)})^{-1}, \quad p_*^{(3)} = G_{13}, \quad p_*^{(4)} = G_{12} \tag{4.22}$$

and the previously unknown formula

$$p_*^{(5)} = (G_{13}J_z + G_{12}J_y)/(J_z + J_y) \tag{4.23}$$

corresponding to a purely torsional FLS, described by the function

$$\varphi = c_1 + c_2 x$$

(c_1 and c_2 are constants of integration), follows from equality (4.21) with the condition that $\varphi \neq 0$. If $G_{13} = G_{12} = G$, formula (4.23) takes the form

$$p_*^{(5)} = G$$

Consequently, in the case of a column made of an isotropic material,

$$p_*^{(3)} = p_*^{(4)} = p_*^{(5)}$$

Note that formula (4.23) and the FLS corresponding to it are completely analogous to the results of the solution of a problem considered earlier in Ref. 6 for a cylindrical shell in the case of its axial compression by a stress $\sigma_{11}^0 = -p$.

4.3. Compression of a column in transverse directions by stresses $\sigma_{22}^0 = -q_2$, $\sigma_{33}^0 = -q_3$

In the case being considered, Eqs (4.15)–(4.17) take the form

$$D_2\psi'' - G_{13}(w' + \psi) + q_3\psi = 0, \quad w'' + \psi' = 0 \tag{4.24}$$

$$D_1\chi'' - G_{12}(\chi - v') + q_2\chi = 0, \quad v'' - \chi' = 0 \tag{4.25}$$

$$B_p\varphi'' + (q_2 + q_3)F\varphi = 0 \tag{4.26}$$

Problems described by equations of the type of (4.24) and (4.25) are completely analogous to the problem of the planar FLS of a strip-beam under transverse compression which was investigated earlier in Ref. 5. In the case of zero variability of the functions appearing in these equations, their non-zero solutions $\psi = \text{const}$, $w = 0$ and $\chi = \text{const}$, $v = 0$ are possible for the bifurcation values

$$q_3^{(1)} = G_{13}, \quad q_2^{(1)} = G_{12} \tag{4.27}$$

which correspond to purely shear FLS. The other bifurcation values of the stresses

$$q_3^{(2)} = D_2\pi^2/a^2, \quad q_2^{(2)} = D_1\pi^2/a^2 \tag{4.28}$$

correspond to purely by flexural FLS. These formula are completely analogous to the formulae of the classical problem of the stability of a column under axial compression by a force which is numerically equal to $pF = q_3^{(2)}F$ and $pF = q_2^{(2)}F$ respectively.

If the load parameter $q = q_2 + q_3$ is introduced into the treatment, the previously unknown formula

$$q_* = B_p\pi^2/(Fa)^2 = (G_{13}J_z + G_{12}J_y)\pi^2/(Fa^2) \tag{4.29}$$

for its bifurcation value follows from Eq. (4.26). This formula corresponds to a purely torsional FLS of the column, which is described by the functions $\varphi = \varphi_0\sin(\pi x/a)$ or $\varphi = \varphi_0\cos(\pi x/a)$ and is completely analogous to the torsional FLS of a cylindrical shell under the uniform external pressure which was investigated earlier in Ref. 6. The other FLS of such a shell when, in the case of an external pressure, it is transformed after loss of stability into a cylindrical shell with parallel sloping cuts,[6] is analogous to the purely shear FLS of a column established above with bifurcation values which are determined using formulae (4.27).

When $G_{13} = G_{12} = G$, formula (4.29) has been established earlier[17] in the case when only a load $q_2 = 0$ acts and $q_3 = 0$.

Introducing the parameter $r = q_2/q_3$, we rewrite formula (4.29) in the form

$$q_2^{(3)} = (G_{13}J_z + G_{12}J_y)\pi^2/\lfloor Fa^2(1 + r)\rfloor$$

whence it follows that, when $r = -1$ (that is, when the column is compressed in one of the transverse directions and stretched in the other direction by equal forces),we have $q_2^{(3)} = \infty$ (there is no loss of stability by a torsional FLS) and, when $r = 1$, the magnitude of $q_2^{(3)}$ is halved. If, however, $r = 0$, then the inequality $q_2^{(3)} < q_2^{(2)}$ can only be satisfied when $G_{13}J_z = G_{12}J_y < E_1J_z$, which can occur in the case of columns made of composite materials.

4.4. Pure bending of a column by end bending moments $M_z^0 = m$ in the xz plane

By virtue of the conditions $Q_x^0 = M_y^0 = \sigma_{22}^0 = 0$, the system of equations (4.16) in the case being considered only has a trivial solution and, after some elementary reduction, the system of equations (4.15) and (4.17) reduces to the single resolvent

$$E_1J_y\psi''' - \frac{m^2}{B_p}\left(\frac{E_1J_y}{G_{13}F}\psi''' - \psi'\right) = 0 \tag{4.30}$$

from which, in the case of hinged edges $x = 0$ and $x = a$, the smallest bifurcation value of the parameter m

$$m_*^{ft} = \sqrt{E_1J_yB_p\pi^2/[a^2(1+k_{13})]} \tag{4.31}$$

which is obtained when $n = 1$, is established using the representation $\psi = \psi_0\cos(n\pi x/a)$, $n = 1, 2, \ldots$. By virtue of the conditions $\psi \neq 0$, $w \neq 0$, $\varphi \neq 0$, the form of loss of stability of a column corresponding to this formula is of the bending-torsional type.

Note also that, in the case of pure bending of a column in the xy plane by a moment m, it is sufficient to replace the quantity J_y in formula (4.31) by the quantity J_z.

4.5. Axial compression of a column accompanying its simultaneous bending in the xz plane

In the case being considered $Q_x^0 == pF$, $M_z^0 = m$, and the system of equations therefore decomposes into two unconnected systems: (4.20) and

$$\begin{gathered} D_2\psi'' - G_{13}(w' + \psi) = 0, \quad G_{13}F(w'' + \psi') - pFw'' + m\varphi'' = 0 \\ (B_p - pJ_p)\varphi'' + mw'' = 0 \end{gathered} \tag{4.32}$$

The first and fourth bifurcation values of the load p, which are determined by the first and fourth formula of (4.22), follow from Eq. (4.20), and Eq. (4.32) reduce to a single resolvent

$$E_1J_y\psi''' - \left(Fp + \frac{m^2}{B_p - pJ_p}\right)\left(\frac{E_1J_y}{G_{13}F}\psi''' + \psi'\right) = 0 \tag{4.33}$$

by which the bending-torsional FLS of the column is described.

If the parameter r_m is introduced in accordance with the relation

$$m = r_mW_yp \tag{4.34}$$

where W_v is the moment of the stiffness of a cross-section of the column with respect to the principal central y axis, then, in the case of a hinged support of the edges $x = 0$ and $x = a$, representing the function ψ in the form $\psi = \psi_0\cos(\pi x/a)$, we arrive at a quadratic equation, the roots of which are

$$p^\pm = (FB_p + J_pP_y \pm \sqrt{(FB_p - J_pP_y)^2 + 4r_m^2W_y^2B_pP_y})/[2(FJ_p - r_m^2W_y^2)] \tag{4.35}$$

where

$$P_y = E_1J_y\pi^2/[a^2(1+k_{13})]$$

for determining the bifurcation values of p.

Investigations showed that the real values of the critical load are determined by the root p^-; it is obvious that $p^- = P_y/P$ when $r_m = 0$.

Table 2

r_m	$E_1/G_{13}=7$ $a/b=10$		$E_1/G_{13}=700$ $a/b=10$		$E_1/G_{13}=700$ $a/b=100$		$E_1/G_{13}=700$ $a/b=100$	
	$h/b=1$	1/4	1	1/4	1	1/4	1	1/4
0.1	10	0	896	7	0	0	10	0
1	941	7	23666	694	10	0	941	7
2	3562	28	40592	2649	38	0	3562	28
3	7377	63	51430	5553	86	0	7377	63
10	37131	695	78700	30605	941	7	37131	695

In the case when $G_{12}=G_{13}$, the values $(1-p^{-}F/P_y)\times 10^5$ for a column of rectangular cross-section with the geometrical characteristics

$$J_y = bh^3/12, \quad J_z = hb^3/12, \quad W_y = bh^2/6$$

are presented in Table 2 for $h-b$ and $h=b/4$ and different values of the parameters E_1/G_{13}, a/b, r_m. Note that, in the case being considered, formula (4.34) can be represented in the form

$$m = r_m hP/6, \quad P = pF$$

so that the quantity r_m characterizes the eccentricity of the point of application of the compressive force P with respect to the centre of the rectangle (the case of extracentral compression of a column). The values of $(p^{-}F/P_y)\times 10^5$ for $r_m \le 3$ when $m \le hP/2$ shown in Table 2 are of practical interest since, prior to loss of stability, significant buckling of the column, which has not been taken into account in the equations constructed, is possible at higher values of r_m.

Two basic conclusions follow from an analysis of the results obtained: 1) in the case of the joint action of axial compression by a stress p and pure bending by a moment m, a reduction of the critical value $p_* = p^{-}$ is always observed; 2) a significant and pronounced reduction in the value of p_* when $m \neq 0$ is only possible in the case of columns made of a material with a low shear strength; this effect is amplified in proportion to the reduction in the relative length of the column and the parameter h/b.

4.6. Trilateral compression of a column by stresses p, q_2, q_3

In the case being considered, Eqs (4.15) and (4.16) take the following form: the first equations of system (4.15) and (4.16) are identical to the first equations of systems (4.24) and (4.25), and the second equations are identical to the second equations of systems (4.19) and (4.20).

Eq. (4.17) takes the form

$$(B_p - pJ_p)\varphi'' + qF\varphi = 0, \quad q = q_2 + q_3 = r_q p \tag{4.36}$$

For the boundary conditions $\varphi'(x=0)=\varphi'(x=a)=0$ or $\varphi(x=0)=\varphi(x=a)=0$, the solution of this equation gives the critical value

$$p_* = B_p\pi^2/(J_p\pi^2 + r_qFa^2) \tag{4.37}$$

which corresponds to a purely torsional FLS of the bar.

The system of equations (4.15) and (4.16) reduces to two unconnected resolvents

$$D_2C_2\psi''' + (p+q_3C_2)\psi' = 0, \quad D_1C_1\chi''' + (p+q_2C_1)\chi' = 0; \quad C_i = 1 - p/G_{1(1+i)} \tag{4.38}$$

by which problems concerning bending-shear FLS in the xz and xy planes are described.

We will now consider the solution of the first of these equations in greater detail. In order to do this, we relate the stresses p and q_3 by means of the relations $p+q_3=\tilde{p}$, $q=r_3p$, by virtue of which

$$q_3 = r_3\tilde{p}/(1+r_3), \quad p = \tilde{p}/(1+r_3) \tag{4.39}$$

Using these relations, we represent the first integral of the first equation of (4.38) in the form (c_1 is a constant of integration)

$$\psi'' + k^2\psi = c_1 \tag{4.40}$$

Here,

$$k^2 = \frac{r_3\tilde{p} - (1+r_3)^2\tilde{G}_{13}}{(1+r_3)[\tilde{p} - (1+r_3)G_{13}]D_2}, \quad \tilde{G}_{13} = \frac{(1+r_3)G_{13}}{[\tilde{p} - (1+r_3)G_{13}]D_2} \tag{4.41}$$

The general solution of Eq. (4.40) has the form

$$\psi = c_2\sin kx + c_3\cos kx - c_1/k^2 \tag{4.42}$$

On subjecting this solution to the boundary conditions

$$M_y^*(x=0) = M_y^*(x=a) = 0$$

we obtain the equality $c_2 = 0$, while the characteristic equation

$$k\sin ka = 0 \tag{4.43}$$

which has two forms of solutions, follows from the condition $c_3 \neq 0$.

The solution $k = 0$, when the first formula of (4.41) is used, leads to a bifurcation value of the load $\tilde{p}$

$$\tilde{p}_*^{(1)} = \tilde{p}_*^s = G_{13}(1+r_3)^2/r_3 \tag{4.44}$$

which corresponds to the appearance of related shear forms of neutral equilibrium.

When $r_3 \to 0$, which corresponds to the action of only a force p, it follows from inequalities (4.44) that

$$\lim_{r_3\to 0}\tilde{p}_*^s = p_*^s = G_{13} = p_*^{(3)}$$

that is, we obtain the third of the bifurcation values (4.22) and, when $p = 0$, one of the solutions of Eq. (4.40) yields the bifurcation value

$$q_*^s = G_{13} = q_3^{(1)}$$

which also corresponds to a shear FLS with preservation of the linear axis of the columns.

However, for $r_3 \neq 0$ when $r_3 > 0$, $\tilde{p}_*^s$ is always greater than G_{13}. For example, when $r_3 = 1$, it follows from relation (4.44) that $\tilde{p}_*^s = 4G_{13}$. At the same time, by virtue of the equality $p_*^{(1)} = p_*^s = (1+r_3)G_{13}/r_3$, we have $p_*^{(1)} = 2G_{13}$.

The increase in $\tilde{p}_*^s$ when $r_3 > 0$ is physically explained by the fact that, when they act simultaneously on the column, the compressive stresses mutually oppose the transformation of the initial rectangle into a parallelogram. At the same time, for a given $r_3 < 0$, the inequality $\tilde{p}_*^s > 0$, only holds when $r_3 < -1$.

Second solution: $ka = \pi, 2\pi, 3\pi, \ldots$. Using formula (4.41) and introducing the load parameter m into the treatment using the formulae

$$\tilde{p} = m\alpha, \quad \alpha = D_2\pi^2/a^2$$

we obtain the characteristic equation

$$m^2 - \frac{1+r_3}{r_3}\left(\frac{(1+r_3)}{k_{13}} + n^2\right)m + \frac{(1+r_3)^2n^2}{k_{13}r_3} = 0 \tag{4.45}$$

When $r_3 = 0$ (only a force p is acting), on putting $n = 1$, from Eq. (4.45) we find the smallest bifurcation values of the parameter m

$$m_*^{(1)} = m_*^u = 1/(1+k_{13}) \tag{4.46}$$

which corresponds to the well-known solution of the problem of the loss of stability of a column through a bending-shear form.

When $r_3 = -1$, the solution $m_* \equiv 0$ follows from Eq. (4.45) and the solution $\tilde{p}_*^s \equiv 0$ follows from formula (4.44).

Table 3

r_3	0.1	1	10	100	−10	−100
$k_{13}=10^{-2}$						
m_1	1220	401	1210	10201	−810	−9801
$m_1 \times 10^3$	992	997	999	999	999	999
$k_{13}=1$						
$m_1 \times 10$	226	52	122	−1020	−82	−980
$m_2 \times 10^3$	536	764	991	999	989	999
$k_{13}=10^{-2}$						
$m_1 \times 10^2$	1111	202	111	111	91	99
$m_2 \times 10^3$	11	19	108	914	−89	−985

Hence, in the case of a column which is loaded by the stresses $\sigma_x^0 = -p$, $\sigma_z^0 = p$ (or $\sigma_x^0 = p$, $\sigma_0 = -p$), the critical values of the load parameters will be zero values, which correspond both to shear and bending-shear FLS.

In the case when $r_3 \neq 0$, the two bifurcation values of the load parameter m

$$m_*^{(1,2)} = m_*^{f(1,2)} = \frac{(1+r_3)}{2r_3}\left(\frac{1+r_3}{k_{13}}+n\right)\left(1 \pm \sqrt{1-\frac{4n^2 r_3 k_{13}}{(1+r_3+k_{13}n^2)^2}}\right) \tag{4.47}$$

corresponding to bending-shear FLS of the column are determined by the roots of Eq. (4.45). The values of these roots $m_*^{f(1)} = m_1$ and $m_*^{f(2)} = m_2$, found for different values of the defining parameters k_{13} and r_3, are presented in Table 3.

As would be expected, the values of m_*^f, which are of practical interest, are determined by the second (smaller) root of Eq. (4.47) and have the smallest values when $n=1$. When $r_3>0$ in the case of small values of k_{13}, they are close to the values which are determined using formula (4.46), and, when $k_{13}>1$, transverse shear in the column has a strong effect on the value of m_2. At the same time, it can be seen in Table 3 that, when $r_3<1$, the losses of stability of the column can also be determined by the first root of Eq. (4.45) (for example, by the values of m_1 for $k_{13}=100$ when $r_3=-10$ and $r_3=-100$; they have their smallest values when $n=1$).

5. Equations and problems of the stability of a column based on the use of kinematic relations in the complete quadratic approximation

Starting out from the equation following from Eq. (4.11) when expressions (4.14) have been substituted into them, we will first consider a problem concerning the stability of a column accompanying the formation in it of an initial stress $\sigma_{12}^0 = \tau$. If the remaining components of the initial stresses are equal to zero, the above mentioned equations decompose into two unconnected systems

$$G_{13}(w''+\psi')+\tau\varphi' = 0, \quad E_1 J_y \psi'' - G_{13}F(w'+\psi) = 0, \quad B_p\varphi'' - \tau F w' = 0 \tag{5.1}$$

$$E_1 u'' - \tau\chi' = 0, \quad \upsilon'' - \chi' = 0, \quad E_1 J_z \chi'' + G_{12}F(\upsilon' - \chi) = 0 \tag{5.2}$$

The system of equations (5.1) is reduced by means of simple transformations to a single resolvent of the form

$$G_{13}E_1 J_y \psi^{IV}/G_{12} + \tau^2 F^2[E_1 J_y \psi''/(G_{13}F) - \psi]/B_p = 0 \tag{5.3}$$

In the case of hinged support of the edges $x=0$ and $x=a$, the bifurcation value of the stress τ,

$$\tau_*^{ft} = \pi^2\sqrt{G_{13}E_1 J_y B_p/[G_{12}(1+k_{13})]}/(a^2 F) \tag{5.4}$$

which, by virtue of $\psi \neq 0$, $\omega \neq 0$, $\varphi \neq 0$, corresponds to a bending-torsional FLS of the column, follows from it.

At the same time, it can easily be shown that the second system of systems of equations (5.2) which have been constructed is of absolutely no interest. The reason for this, judging from the results obtained earlier in Refs 5,6 is the insufficient degree of accuracy of relations (4.1) and representation (4.6).

5.1. Neutral equilibrium equations

It follows from the analysis of previous results[5,6] that, in the case of the formation of the initial stresses $\sigma_{12}^0, \sigma_{13}^0$ in a column, instead of the incomplete relations (4.1) and (4.2) it is necessary to use the unsimplified (full) kinematic relations of the form

$$\varepsilon_1 = E_{11} + (E_{12}^2 + E_{13}^2)/2, \ldots, \quad \gamma_{12} = E_{12}(1 + E_{22}) + E_{21}(1 + E_{11}) + E_{13}E_{23}, \ldots \tag{5.5}$$

$$\begin{aligned} \sigma_{11}^* &= \sigma_{11} + \sigma_{12}E_{21} + \sigma_{13}E_{31}, \quad \sigma_{12}^* = \sigma_{11}E_{12} + \sigma_{12}(1 + E_{22}) + \sigma_{13}E_{32} \\ \sigma_{13}^* &= \sigma_{11}E_{13} + \sigma_{12}E_{23} + \sigma_{13}(1 + E_{33}), \ldots \end{aligned} \tag{5.6}$$

and to use the representation

$$\mathbf{U} = (u + z\psi - y\chi)\mathbf{i} + (\upsilon - z\varphi + y\theta_2)\mathbf{j} + (w + y\varphi + z\theta_3)\mathbf{k} \tag{5.7}$$

which corresponds to the Timoshenko model taking account of the transverse strains ε_2 and ε_3, instead of (4.6).

According to the adopted refined model of a column, the quantities $E_{\alpha\beta}$ appearing in relations (5.5) will be determined by the expressions

$$\begin{aligned} &E_{11} = u' + z\psi' - y\chi', \quad E_{12} = \upsilon' - z\varphi' + y\theta_2', \quad E_{13} = w' + y\varphi' + z\theta_3' \\ &E_{21} = -\chi, \quad E_{22} = \theta_2, \quad E_{23} = \varphi, \quad E_{31} = \psi, \quad E_{32} = -\varphi, \quad E_{33} = \theta_3 \end{aligned} \tag{5.8}$$

If it is accepted that $\sigma_{23}^0 = 0$ in the initial state then, in the case being considered, the linearized relations

$$\begin{aligned} &\sigma_{11}^* = \sigma_{11} + \sigma_{12}^0E_{21} + \sigma_{13}^0E_{31}, \quad \sigma_{12}^* = \sigma_{12} + \sigma_{11}^0E_{12} + \sigma_{12}^0E_{22} + \\ &+ \sigma_{13}^0E_{32}, \quad \sigma_{13}^* = \sigma_{13} + \sigma_{11}^0E_{13} + \sigma_{12}^0E_{23} + \sigma_{13}^0E_{33}, \ldots \end{aligned} \tag{5.9}$$

which, on substituting the relations (5.8), take the form

$$\begin{aligned} \sigma_{11}^* &= g_{11}(u' + z\psi' - y\chi') + g_{12}\theta_2 + g_{13}\theta_3 - \sigma_{12}^0\chi + \sigma_{13}^0\psi \\ \sigma_{12}^* &= (G_{12} + \sigma_{11}^0)(\upsilon' - z\varphi' + y\theta_2') - G_{12}\chi + \sigma_{12}^0\theta_2 - \sigma_{13}^0\varphi, \ldots \\ \sigma_{33}^* &= g_{13}(u' + z\psi' - y\chi') + g_{23}\theta_2 + g_{33}\theta_3 + \sigma_{13}^0(w' + y\varphi' + z\theta_3') \end{aligned} \tag{5.10}$$

have to be used instead of relations (4.8).

Then, using relations (5.8) and by virtue of the inequalities $E_{22} \neq 0$, $E_{33} \neq 0$, instead of (4.9) we arrive at the variational equation

$$\begin{aligned} &\int_0^a (Q_x^*\delta u' + M_y^*\delta\psi' + M_z^*\delta\chi' + Q_y^*\delta\upsilon' + Q_z^*\delta w' + M_x^*\delta\varphi' + M_{xy}^*\delta\theta_2' + M_{xz}^*\delta\theta_3' + \\ &+ N_z^*\delta\psi + N_y^*\delta\chi + N_x^*\delta\varphi + T_y^*\delta\theta_2 + T_z^*\delta\theta_3)dx = 0 \end{aligned} \tag{5.11}$$

in which, in addition to (4.10), the notation

$$M_{xy}^* = \iint \sigma_{12}^* y dF, \; M_{xz}^* = \iint \sigma_{13}^* z dF, \quad T_y^* = \iint \sigma_{22}^* dF, \quad T_z^* = \iint \sigma_{33}^* dF \tag{5.12}$$

has been introduced.

The six differential equations (4.11), the boundary conditions (4.12) and the two additional equations

$$\frac{dM_{xy}^*}{dx} - T_y^* = 0, \quad \frac{dM_{xz}^*}{dx} - T_z^* = 0 \tag{5.13}$$

and the boundary conditions corresponding to them

$$M_{xy}^* = 0 \text{ When } \delta\theta_2 \neq 0, \quad M_{xz}^* = 0 \text{ When } \delta\theta_3 = 0 \tag{5.14}$$

follow from Eq. (5.11).

The required relations of the form (4.14) for the stresses and moments appearing in Eqs (4.11) and (5.13) are obtained by substituting expressions (5.10) into formulae (4.10) and (5.12):

$$\begin{aligned} Q_x^* &= F(g_{11}u' + g_{12}\theta_2 + g_{13}\theta_3) - Q_y^0\chi + Q_z^0\psi \\ Q_y^* &= (G_{12}F + Q_x^0)\upsilon' - G_{12}F\chi + Q_y^0\theta_2 - Q_z^0\varphi + M_z^0\theta_2' + M_y^0\varphi' \\ Q_z^* &= (G_{13}F + Q_x^0)w' + G_{13}F\psi + M_y^0\theta_3' + M_z^0\varphi' + Q_y^0\varphi + Q_z^0\theta_3 \end{aligned} \tag{5.15}$$

$$\begin{aligned} M_y^* &= g_{11}J_y\psi' - \chi\iint\sigma_{12}^0 z dF + \underline{\psi\iint\sigma_{13}^0 z dF}, \dots \\ M_{xy}^* &= M_z^0\upsilon' + G_{12}J_z\theta_2' + \underline{\theta_2\iint\sigma_{12}^0 y dF} - \varphi\iint\sigma_{13}^0 y dF - \\ &- \underline{\varphi'\iint\sigma_{11}^0 zy dF} + \theta_2'\iint\sigma_{11}^0 y^2 dF \end{aligned} \tag{5.16}$$

The notation for the stresses and moments of the initial state of the column

$$\begin{aligned} &Q_x^0 = \iint\sigma_{11}^0 dF, \quad M_y^0 = \iint\sigma_{11}^0 z dF, \quad M_z^0 = \iint\sigma_{11}^0 y dF \\ &T_y^0 = \iint\sigma_{22}^0 dF, \quad T_z^0 = \iint\sigma_{33}^0 dF, \quad Q_y^0 = \iint\sigma_{12}^0 dF, \quad Q_z^0 = \iint\sigma_{13}^0 dF \end{aligned} \tag{5.17}$$

has been introduced here.

The elasticity relations of the form (5.16) which have been constructed contain a number of unimportant terms that can be discarded without any loss of richness of content of the problems which are formulated on their basis. In order to carry out such simplifications, while remaining within the framework of the degree of accuracy of the relations in the classical theory of columns, we will assume that it is permissible to determine the normal stresses σ_{11}^0 using formula (4.13) and the shear stresses σ_{12}^0 and σ_{13}^0 using the formulae

$$\sigma_{12}^0 = Q_y^0/F + zM_{\text{tor}}^0/J_p, \quad \sigma_{13}^0 = Q_z^0/F - yM_{\text{tor}}^0/J_p \tag{5.18}$$

where M_{tor}^0 and Q_y^0, Q_z^0 are the torsional moment and the shearing forces in the cross-sections of the column in its initial state.

When formulae (4.13) and (5.18) are used for quantities of the initial state appearing in relations (5.16), in addition to expressions (5.17) we have the exact formulae

$$\begin{aligned} &\iint\sigma_{12}^0 z dF = J_y M_{\text{tor}}^2/J_p, \quad \iint\sigma_{13}^0 z dF = 0, \quad \iint\sigma_{12}^0 y dF = 0 \\ &\iint\sigma_{13}^0 y dF = -J_z M_{\text{tor}}^2/J_p, \quad \iint\sigma_{11}^0 z^2 dF = J_y Q_x^0/F, \quad \iint\sigma_{11}^0 y^2 dF = J_z Q_x^0/F \\ &\iint\sigma_{11}^0(y^2 + z^2)dF = J_p Q_x^0/F, \quad \iint(\sigma_{12}^0 y + \sigma_{13}^0 z)dF = 0 \\ &\iint\sigma_{11}^0 z^2 dF = J_y Q_x^0/F, \quad \iint\sigma_{11}^0 y^2 dF = J_z Q_x^0/F \end{aligned} \tag{5.19}$$

and we also assume that $\int\int z3dF = 0$, $\int\int y3dF = 0$, $\int\int\sigma_{11}^0 zydF = 0$.

As a result, instead of relations (5.16), we arrive at simplified relations of the form

$$\begin{aligned} M_y^* &= g_{11}J_y\psi' - J_y M_{\text{tor}}^0\chi/J_p, \quad M_z^* = g_{11}J_z\chi' + J_z M_{\text{tor}}^2\psi/J_p \\ M_x^* &= (B_p + J_p Q_x^0/F)\varphi' + M_z^0 w' - M_y^0\upsilon' - J_z M_{\text{tor}}^2\theta_3/J_p - J_y M_{\text{tor}}^0\theta_2/J_p \\ N_z^* &= (G_{13}F + T_z^0)\psi + G_{13}Fw' + Q_z^0 u' + J_z M_{\text{tor}}^0\chi'/J_p \\ N_y^* &= (G_{12}F + T_y^0)\chi - G_{12}F\upsilon' - Q_y^0 u' - J_y M_{\text{tor}}^2\psi'/J_p \\ N_x^* &= (T_y^0 + T_z^0)\varphi + Q_y^0 w' - Q_z^0\upsilon' + J_y M_{\text{tor}}^2\theta_3'/J_p + J_z M_{\text{tor}}^2\theta_2'/J_p \\ T_y^* &= F(g_{12}u' + g_{22}\theta_2 + g_{23}\theta_3) + Q_y^0\upsilon' - J_y M_{\text{tor}}^2\varphi'/J_p \\ T_z^* &= F(g_{13}u' + g_{23}\theta_2 + g_{33}\theta_3) + Q_z^0 w' - J_z M_{\text{tor}}^2\varphi'/J_p \\ M_{xz}^* &= G_{13}J_y\theta_3' + M_y w' + J_y M_{\text{tor}}^2\varphi/J_p + J_y Q_x^0\theta_3'/F \\ M_{xy}^* &= G_{12}J_z\theta_2' + M_z\upsilon' + J_z M_{\text{tor}}^2\varphi/J_p + J_z Q_x^0\theta_2'/F \end{aligned} \tag{5.20}$$

In order to carry out further analysis and a possible simplification of relations (5.15) and (5.20) we return to the non-linear kinematic relations (5.5). Introducing expressions (5.8) into them, for ε_1, ε_2 and ε_3 we obtain the relations (for ε_1 with an accuracy up to $z^2(0)$, $y^2(0)$, $zy(0)$ which is customarily accepted in the theory of columns, plates and shells)

$$\begin{aligned}\varepsilon_1 &= u' + (\upsilon'^2 + w'^2)/2 + z(\psi' - \upsilon'\varphi' + w'\theta_3') - y(\chi' - \upsilon'\theta_2' - w'\varphi')\\ \varepsilon_2 &= \theta_2 + (\chi^2 + \varphi^2)/2, \quad \varepsilon_3 = \theta_3 + (\psi^2 + \varphi^2)/2\end{aligned} \tag{5.21}$$

and the relations

$$\begin{aligned}\gamma_{12} &= \upsilon' - \chi + \theta_2\upsilon' - \chi u' + \varphi w' - z(\varphi' + \theta_2\varphi' + \chi\psi' - \varphi\theta_3') + \tilde{\gamma}_{12}\\ \gamma_{13} &= w' + \psi + \theta_3 w' + \psi u' - \varphi\upsilon' + y(\varphi' + \theta_3\varphi' - \psi\chi' - \varphi\theta_2') + \tilde{\gamma}_{13}\\ \tilde{\gamma}_{12} &= y(\theta_2' + \theta_2\theta_2' + \chi\chi' + \varphi\varphi'), \quad \tilde{\gamma}_{13} = z(\theta_3' + \theta_3\theta_3' + \psi\psi' + \varphi\varphi')\end{aligned} \tag{5.22}$$

for the shear strains.

It is seen that the equalities $\tilde{\gamma}_{12} = y\varepsilon_2'$ and $\tilde{\gamma}_{13} = z\varepsilon_3'$ hold for the terms $\tilde{\gamma}_{12}$ and $\tilde{\gamma}_{13}$ if the formulae

$$\varepsilon_2 = \theta_2 + (\theta_2^2 + \chi^2 + \varphi^2)/2, \quad \varepsilon_3 = \theta_3 + (\theta_3^2 + \psi^2 + \varphi^2)/2$$

are used to calculate ε_2 and ε_3 which, as has been shown in Refs 3,4, are inconsistent and incorrect. In the theory of thin shells, terms of these forms are usually neglected.[8,9]

As a result of the simplifications introduced above, we arrive at the following expression (putting $\sigma_{23} \approx 0$) for the variation in the potential energy of the deformation of the column

$$\begin{aligned}\delta U &= \iint\limits_V (\sigma_{11}\delta\varepsilon_1 + \sigma_{22}\delta\varepsilon_2 + \delta_{33}\delta\varepsilon_3 + \sigma_{12}\delta\gamma_{12} + \sigma_{13}\delta\gamma_{13})dV =\\ &= \int_0^a (Q_x^*\delta u' + M_y^*\delta\psi' + M_z^*\delta\chi' + Q_y^*\delta\upsilon' + Q_z^*\delta w' + M_x\delta\varphi' +\\ &+ M_{xy}^*\delta\theta_2' + M_{xz}^*\delta\theta_3' + N_z^*\delta\psi + N_y^*\delta\chi + N_x^*\delta\varphi + T_y^*\delta\theta_2 + T_z^*\delta\theta_3)dx\end{aligned} \tag{5.23}$$

where the notation

$$\begin{aligned}Q_x^* &= \iint(\sigma_{11} - \sigma_{12}\chi + \sigma_{13}\psi)dF, \ldots\\ T_y^* &= \iint(\sigma_{12}\upsilon' - \sigma_{13}\varphi' z + \sigma_{22})dF, \quad T_z^* = \iint(\sigma_{33} + \sigma_{13}w' + \sigma_{13}\varphi' y)dF\end{aligned} \tag{5.24}$$

has been introduced.

We linearize relations (5.24) in the neighbourhood of the initial stress state, prescribed by the stresses Q_x^0, Q_y^0, Q_z^0, T_y^0, T_z^0 and the moments M_y^0, M_z^0, M_{tor}^0. Then, when account is taken of formulae (4.13), (5.18) and (5.19) for Q_x^*, Q_y^*, Q_z^*, M_y^*, M_z^*, T_y^*, T_z^*, we obtain the same expressions which are contained in equalities (5.15) and (5.20), and simplified expressions of the form

$$\begin{aligned}M_x^* &= B_p\varphi' - M_y^0\upsilon' + M_z^0 w' - M_{\text{tor}}^2(J_z\theta_3 + J_y\theta_2)/J_p\\ N_z^* &= G_{13}F(w' + \psi) + T_z^0\psi + J_z M_{\text{tor}}^2\chi'/J_p, \quad N_y^* = G_{12}F(\chi - \upsilon') + T_y^0\chi - J_y M_{\text{tor}}^2\psi'/J_p\\ N_x^* &= Q_y^0 w' - Q_z^0\upsilon' + (T_y^0 + T_z^0)\varphi, \quad M_{xy}^* = M_z^0\upsilon' + J_z M_{\text{tor}}^2\varphi/J_p, \quad M_{xz}^* = M_y^0 w' + J_y M_{\text{tor}}^2\varphi/J_p\end{aligned} \tag{5.25}$$

are obtained for the other stresses and moments characterizing the neutral equilibrium of the column.

It is seen that in the expression for M_x^* according to the first formula of (5.25) and unlike in the second expression of (5.20), the term $J_p Q_x^0\varphi'/F$ is missing. On account of this, the solutions for p_* of the corresponding equations constructed from (4.12) and (5.13), when the simplified relations are used in the case of axial compression of a column, only reduce to formulae (4.22) since the torsional FLS of the bar accompanying its axial compression is not described by them. At the same time, the simplified expressions derived in this section turn out to be more interesting than the relations in Section 4 in the case of other forms of the initial stress state. The solutions of a number of new non-classical problems of the stability of a column, which are constructed below, serve as a confirmation of what has been said.

5.2. Pure bending of a column by end bending moments $M_z^0 = M$

In the case being considered, the system of equations (4.11), (5.13) decomposes, when the corresponding relations from (5.15), (5.20) and (5.25) are used, into the two unconnected systems

$$g_{11}J_y\psi'' - G_{13}F(w' + \psi) = 0, \quad G_{13}F(w'' + \psi') + m\varphi'' = 0, \quad B_p\varphi'' + mw'' = 0 \tag{5.26}$$

$$\begin{gathered} g_{11}u'' + g_{12}\theta_2' + g_{13}\theta_3' = 0, \quad G_{12}F(\upsilon'' - \chi') + m\theta_2'' = 0, \quad g_{11}J_z\chi'' - G_{12}F(\chi - \upsilon') = 0 \\ m\upsilon'' - F(g_{12}u' + g_{22}\theta_2 + g_{23}\theta_3) = 0, \quad g_{13}u' + g_{23}\theta_2 + g_{33}\theta_3 = 0 \end{gathered} \tag{5.27}$$

The first equations of systems (4.32) and (5.26) only differ in the coefficient g_{11} in (5.26) instead of E_1 in (4.32), the bending-torsional FLS of the column is described by them and the first, fourth and fifth equations of system (5.27) enable us to establish the relations (c_1 is a constant of integration)

$$u' = c_1 - \frac{\upsilon_{12}}{E_2F}m\upsilon'', \quad \theta_2 = -\upsilon_{21}c_1 + \frac{m}{E_2F}\upsilon'', \quad \theta_3 = -\upsilon_{31}c_1 - \frac{\upsilon_{32}}{E_2F}m\upsilon'' \tag{5.28}$$

When they are used, the remaining two equations of system (5.27) reduce to a single resolvent (c_2 is a constant of integration)

$$g_{11}J_z\upsilon''' - m^2\lfloor \upsilon''' - g_{11}J_z\upsilon^V/(G_{12}F)\rfloor/(E_2F) = c_2 \tag{5.29}$$

by which the bending-shear FLS of a column is described. In the case of the hinged support of its edges, the formula

$$m_*^{fs} = \sqrt{g_{11}J_zE_2F/(1 + k_{12}^*n^2)}, \quad k_{12}^* = g_{11}J_z/(G_{12}F) \approx k_{12}, \quad n = 1, 2, \ldots \tag{5.30}$$

which reduces, when $n \to \infty$, to the value $m_*^{fs} = 0$, follows from Eq. (5.29) for determining the bifurcation values of the bending moment.

In content, this result is analogous to the result obtained in Ref. 18 for a three-layer beam in the case of pure bending. However, there are fundamental differences between them: in a three-layer beam $m_*^{\min}$ has a finite positive value, which is interpreted as the work done in the bending of a separate compressed external layer accompanying the loss of stability of the beam as of a whole and, in the case being considered, the hypothetical external layers have a zero characteristic bending stiffness in a homogeneous beam.

Hence, in the case of the pure bending of a column, only the realization of the bending-torsional FLS established in Subsection 4.4 is possible and this is accompanied by torsion and bending in a direction perpendicular to the plane of action of the external bending moment.

It should be noted that, in the treatment of the joint action on a column of an axial compression p and a bending moment $M_z^0 = m$, the system of equations (4.11), (5.13), when relations (5.15), (5.20) and (5.25) are used, also splits into two unconnected systems of equations. The first of these is identical to the system of equations (4.8) if g_{11} is replaced by E_1 and the bending-shear FLS, which leads to the same trivial result obtained above in the form of formula (5.30), will be described by the second system.

5.3. Pure torsion of a column with a moment $M_{tor}^0 = m$

In the case being considered, the system of equations (4.11), (5.13), when the corresponding relations from (5.15), (5.20) and (5.25) are used, also splits into two unconnected systems of equations

$$\begin{gathered} \upsilon'' - \chi' = 0, \quad w'' + \psi' = 0, \quad g_{11}J_y\psi'' - G_{13}F(w' + \psi) - M\chi' = 0 \\ g_{11}J_z\chi'' - G_{12}F(\chi - \upsilon') + M\psi' = 0 \end{gathered} \tag{5.31}$$

$$\begin{gathered} g_{11}u'' + g_{12}\theta_2' + g_{13}\theta_3' = 0, \quad B_p\varphi'' - M(\theta_2' + \theta_3') = 0 \\ M\varphi' - F(g_{12}u' + g_{22}\theta_2 + g_{23}\theta_3) = 0 \\ M\varphi' - F(g_{13}u' + g_{23}\theta_2 + g_{33}\theta_3) = 0 \end{gathered} \tag{5.32}$$

of which the first can be reduced to a single resolvent of the form

$$\upsilon^{IV} + M^2\upsilon''/(g_{11}^2 J_z J_y) = 0 \tag{5.33}$$

In the case of a hinged support of the edges, sections, the bifurcation value of the torsional moment

$$M_{(1)}^* = g_{11}\pi\sqrt{J_z J_y}/a \tag{5.34}$$

which corresponds to a purely flexural FLS of the column, follows from Eq. (5.33).

By analogy with relations (5.28), the relations (c_1 is a constant of integration)

$$\begin{gathered} u' = -\frac{M}{F}\left(\frac{\upsilon_{12}}{E_2} + \frac{\upsilon_{13}}{E_3}\right)\varphi' + \frac{c_1}{E_1}, \quad \theta_2 = \frac{M}{F}\frac{(E_3 - E_2\upsilon_{23})}{E_2E_3}\varphi' - \frac{\upsilon_{21}}{E_1}c_1 \\ \theta_3 = \frac{M}{F}\frac{(E_2 - \upsilon_{32}E_3)}{E_2E_3}\varphi' - \frac{\upsilon_{31}}{E_1}c_1 \end{gathered} \tag{5.35}$$

are determined from the first and the last two equations of system (5.32). When they are used, the second equation of the system is transformed to the form

$$[B_p - M^2(E_2 + E_3 - E_2\upsilon_{23} - E_3\upsilon_{32})/(FE_2E_3)]\varphi'' = 0 \tag{5.36}$$

and the second bifurcation value of the torsional moment

$$M_{(2)}^* = \sqrt{B_p F E_2 E_3/[E_2(1 - \upsilon_{23}) + E_3(1 - \upsilon_{32})]} \tag{5.37}$$

which corresponds to a purely torsional FLS of the bar, follows from this.

Note that the values of $M_{(1)}^*$ and $M_{(2)}^*$ which have been found are of the orders $M_{(1)}^* \approx E_1 a^3 \varepsilon^4$, $M_{(2)}^* \approx E_1 a^3 \varepsilon^3 \sqrt{G/E_2}$ if $E_2 \approx E_3$, $b \approx a\varepsilon$, $h \approx a\varepsilon$, $\varepsilon \ll 1$. Consequently, $M_{(1)}^* \approx M_{(2)}^*$ if $\sqrt{G/E_2} \approx \varepsilon$.

5.4. A column under conditions of pure shear

We will assume that, in the column, $Q_z^0 \neq 0$ and the remaining internal stresses and moments of the initial state are equal to zero. In the case being considered, the system of equations (4.11), (5.13), when the corresponding relations from (5.15), (5.20) and (5.25) are used, splits, as in the previous case, into two systems of equations

$$\begin{gathered} F(g_{11}u' + g_{12}\theta_2 + g_{13}\theta_3)' + Q_z^0\psi' = 0, \quad g_{12}u' + g_{22}\theta_2 + g_{23}\theta_3 = 0 \\ F(g_{13}u' + g_{23}\theta_2 + g_{33}\theta_3)' + Q_z^0 w' = 0 \end{gathered} \tag{5.38}$$

$$\begin{gathered} G_{13}F(w' + \psi)' + Q_z^0\theta_3 = 0, \quad g_{11}J_y\psi'' - G_{13}F(w' + \psi) = 0 \\ G_{12}F(\upsilon' - \chi)' - Q_z^0\varphi' = 0, \quad g_{11}J_z\chi'' - G_{13}F(\chi - \upsilon') = 0 \\ B_p\varphi'' + Q_z^0\upsilon' = 0 \end{gathered} \tag{5.39}$$

The FLS of the bar is described by the first system (5.38) which has been thoroughly investigated[5] in the case of a column in the form of a strip and the second system (5.39), consisting of three equations, reduces to of a single resolvent of the form

$$g_{11}J_z B_p \varphi^{IV} + (Q_z^0)^2(k_{12}^*\varphi''' + \varphi') = 0 \tag{5.40}$$

In the case of hinged support of the edges of the column, the bifurcation value

$$Q_z^* = \pi^2\sqrt{g_{11}J_z(G_{13}J_z + G_{12}J_y)/(1 + k_{12}^*)/a^2} \tag{5.41}$$

which corresponds to a previously unknown bending-torsional FLS of the column, follows from this.

It can be shown that, in the case of real columns, the value of Q_z^*, determined using formula (5.41), is much smaller than the value of Q_z^*, determined by the solution of Eq. (5.38).

6. Classical and non-classical forms of loss of stability of an orthotropic cylindrical shell under torsion

6.1. Formulation of the problem

We shall consider a cylindrical shell of thickness $t=2h$, with a radius of the middle surface R and a length L made of an orthotropic material with the elastic characteristics E_1, E_2, G_{12}, υ_{12}, $\upsilon_{21}=\upsilon_{12}E_2/E_1$ and which is loaded at the edges $x=0$ and $x=L$ by a torsional moment M. We will assume that, in the initial (subcritical) state of the shell, the stress state which is formed in it has a zero moment and is solely characterized by the single non-zero value of the linear shear force $T_{12}^0=S=M/(2\pi R^2)$. By virtue of the relations

$$A_1 = 1, \quad A_2 = R, \quad k_1 = 0, \quad k_2 = 1/R$$

the linearized neutral equilibrium equations of general form constructed in Section 2 can be represented in the case being considered in the form

$$\begin{gathered} R\partial_x S_{11} + \partial_\theta S_{21} = 0, \quad R\partial_x S_{12} + \partial_\theta S_{22} + S_{23} = 0, \quad R\partial_x S_{13} + \partial_\theta S_{23} - S_{22} = 0 \\ R\partial_x M_{11} + \partial_\theta M_{21} - RT_{13} = 0, \quad R\partial_x M_{12} + \partial_\theta M_{22} - RT_{23} = 0 \end{gathered} \tag{6.1}$$

Here ∂_x and ∂_θ are differential operators with respect to the axial and angular coordinates x and θ; a similar notation will henceforth be used for differential operators.

The last two equations of system (6.1) are relations between the shearing forces T_{13} and T_{23} and the internal bending and twisting moments $M_{11}, M_{22}, M_{12}=M_{21}$. According to the Kirchhoff-Love model, when $\gamma_i=-\omega_i$ and with displacements $u_1=u$, $u_2=\upsilon$, w of the points of the middle surface, the latter are consented by the elasticity relations

$$\begin{gathered} M_{11} = -D_{11}\left(\partial_{xx}w + \upsilon_{21}\frac{\partial_{\theta\theta}w - \partial_\theta \upsilon}{R^2}\right), \quad M_{22} = -D_{22}\left(\frac{\partial_{\theta\theta}w - \partial_\theta w}{R^2} + \upsilon_{12}\partial_{xx}w\right) \\ M_{12} = M_{21} = -D_{12}\frac{2\partial_{x\theta}w - \partial_x \upsilon}{R} \end{gathered} \tag{6.2}$$

where

$$D_{ij} = t^2 B_{ij}/12, \quad B_{12} = G_{12}t, \quad B_{ii} = E_i t/(1-\upsilon_{12}\upsilon_{21})$$

The relations

$$\begin{gathered} S_{11} = B_{11}\left(\partial_x u + \upsilon_{21}\frac{\partial_\theta \upsilon + w}{R}\right) + \underline{S\frac{\partial_\theta u}{R}}, \quad S_{22} = B_{22}\left(\frac{\partial_\theta \upsilon + w}{R} + \upsilon_{12}\partial_x u\right) + \underline{S\partial_x \upsilon} \\ S_{12} = B_{12}\left(\partial_x \upsilon + \frac{\partial_\theta u}{R}\right) + \underline{S\frac{\partial_\theta \upsilon + w}{R}}, \quad S_{21} = B_{12}\left(\partial_x \upsilon + \frac{\partial_\theta u}{R}\right) + \underline{S\partial_x u} \\ S_{13} = T_{13} + S\frac{\partial_x w - \upsilon}{R}, \quad S_{23} = T_{23} + S\partial_x w \end{gathered} \tag{6.3}$$

hold for the remaining internal stresses occurring in system (6.1).

System (6.1)–(6.3) differs from the known equations of the theory of the mean flexure of shells by the presence of the underlined terms in relations (6.3) and, due to these terms, one of the possible forms of loss of stability of a cylindrical shell under torsion was found in Ref. 6 within the framework of the equations of membrane theory when the assumptions

$$T_{13} = T_{23} = M_{11} = M_{12} = M_{22} = 0$$

were made. It is purely torsional and a bifurcation value of the stress S

$$S_{(1)}^* = \sqrt{B_2 B_{12}}/2, \quad B_2 = B_{22}(1-\upsilon_{12}\upsilon_{21}) = E_2 t \tag{6.4}$$

corresponds to it.

It will be shown below that, besides the above mentioned FLS and the two classical FLS, which have been thoroughly studied up to the present time in the case of isotropic shells,[19] new FLS which are unknown in the literature are also described by the system of equations which has been constructed. To investigate these, we reduce the system constructed to a system of three equations in the displacements u, v, and w

$$\begin{aligned}
&f_1 = L_{11}u + L_{12}v + v_{21}\partial_\zeta w = 0, \quad f_2 = L_{21}u + (1 + c^2 t)L_{22}v + \\
&+ \{\partial_\theta + T\partial_\zeta - c^2\partial_\theta[(v_{12} + 2g_2)\partial_{\zeta\zeta} + \partial_{\theta\theta}]\}w = 0 \\
&f_3 = v_{12}\partial_\zeta u + \{\partial_\theta + T\partial_\zeta - c^2\partial_\theta[(v_{12} + 2g_2)\partial_{\zeta\zeta} + \partial_{\theta\theta}]\}v + \\
&+ \{1 - T\partial_{\zeta\theta} + c^2[\varepsilon^{-1}\partial_{\zeta\zeta\zeta\zeta} + 2(v_{12} + 2g_2)\partial_{\zeta\zeta\theta\theta} + \partial_{\theta\theta\theta\theta}]\}w = 0
\end{aligned} \tag{6.5}$$

where

$$\begin{aligned}
&L_{11} = \partial_{\zeta\zeta} + g_1\partial_{\theta\theta} + \tilde{\varepsilon}T\partial_{\zeta\theta}, \quad L_{12} = (v_{21} + g_1)\partial_{\zeta\theta} \\
&L_{21} = (v_{12} + g_2)\partial_{\zeta\theta}, \quad L_{22} = g_2\partial_{\zeta\zeta} + \partial_{\theta\theta} + T\partial_{\zeta\theta}
\end{aligned} \tag{6.6}$$

$$g_i = \frac{G_{12}}{B_{ii}}, \quad i = 1, 2, \quad \tilde{\varepsilon} = \frac{E_2}{E_1}, \quad \varsigma = \frac{x}{R}, \quad c^2 = \frac{t^2}{12R^2}, \quad T = \frac{2S}{B_{22}} \tag{6.7}$$

If we put $T = 0$ and $c = 0$ in the first two equations of system (6.5), we arrive at the neutral equilibrium equations of the classical theory of shallow shells, on which the investigations into the stability of cylindrical shells under torsion that are known in the literature are mainly based.

6.2. Forms of loss of stability described by the equations of membrane theory

When $c^2 = 0$, which corresponds to the introduction of the assumptions $T_{13} = T_{23} = 0$ of membrane theory, the system of equations (6.5) can be represented in the form

$$\begin{aligned}
&L_{11}u + L_{12}v + v_{21}\partial_\varsigma w = 0, \quad L_{21}u + L_{22}v + \partial_\theta w + T\partial_\varsigma w = 0 \\
&v_{12}\partial_\varsigma u + \partial_\theta v + T\partial_\varsigma v + w - T\partial_{\varsigma\theta}w = 0
\end{aligned} \tag{6.8}$$

The first two equations of system (6.8) enable us to construct the following differential relations

$$\begin{aligned}
&Du = \tilde{\varepsilon}\partial_\zeta(\partial_{\theta\theta} - \partial_{\zeta\zeta} + T\partial_{\zeta\theta})w \\
&Dv = -\partial_\theta[g_2^{-1}(1 - v_{12}v_{21} - v_{21}g_2)\partial_{\zeta\zeta} + \tilde{\varepsilon}\partial_{\theta\theta}]w - \\
&- Tg_2^{-1}\partial_\zeta[\partial_{\zeta\zeta} + (\tilde{\varepsilon} + g_1)\partial_{\theta\theta} + \varepsilon T\partial_{\zeta\theta}]w
\end{aligned} \tag{6.9}$$

where

$$\begin{aligned}
&D = \partial_{\zeta\zeta\zeta\zeta} + g_2^{-1}(1 - v_{12}v_{21} - 2v_{12}g_1)\partial_{\zeta\zeta\theta\theta} + \tilde{\varepsilon}\partial_{\theta\theta\theta\theta} + \\
&+ Tg_2^{-1}\partial_{\zeta\theta}[(1 + g_1)\partial_{\zeta\zeta} + (\tilde{\varepsilon} + g_1)\partial_{\theta\theta} + \tilde{\varepsilon}T\partial_{\zeta\theta}]
\end{aligned}$$

On applying the operator D to the third equation of system (6.8) and taking account of the expressions for the operators (6.6), after some reduction we obtain the resolvent

$$N(w) = 0 \tag{6.10}$$

where

$$\begin{aligned}
&N(w) = (1 - v_{12}v_{21})\partial_{\zeta\zeta\zeta\zeta}w - Tg_2^{-1}\{(\partial_{\zeta\zeta} - \tilde{\varepsilon}) + \\
&+ (1 + \partial_{\theta\theta})[g_1\partial_{\theta\theta} + (1 - v_{12}v_{21} - 2v_{12}g_1)\partial_{\zeta\zeta}\}\partial_{\zeta\theta}w - \\
&- T^2\{\tilde{\varepsilon}\partial_{\zeta\zeta\theta\theta} + g_2^{-1}(1 + \partial_{\theta\theta})[\partial_{\zeta\zeta} + (\tilde{\varepsilon} + g_1)\partial_{\theta\theta}]\}\partial_{\zeta\zeta}w - \tilde{\varepsilon}T^3g_2^{-1}(1 + \partial_{\theta\theta})\partial_{\zeta\zeta\zeta\theta}w
\end{aligned} \tag{6.11}$$

It can be shown that, according to relations (6.9), equations of the form

$$N(u) = 0, \quad N(v) = 0 \tag{6.12}$$

in which the variables u and v are the unknowns, also hold.

If it is assumed that the function w has zero variability in the peripheral direction θ, that is, $w = w(\theta)$, then Eq. (6.10) takes the form

$$(1-\upsilon_{12}\upsilon_{21}-T^2/g_2)\partial_{\zeta\zeta\zeta\zeta}w = 0 \tag{6.13}$$

Subject to the condition$\partial_{\zeta\zeta\zeta\zeta}w \neq 0$, the bifurcation value of the load parameter$T^*_{(1)} = \sqrt{g_2(1-\upsilon_{12}\upsilon_{21})}$, which corresponds to formula (6.4) which was derived earlier, also follows from this equation.

We will now adopt the representation

$$w = W_1\sin(\pi R/L-\theta)+W_2\cos(\pi R/L-\theta) \tag{6.14}$$

for the function w, which corresponds to the formation of a single half-wave along the generatrix and a single half-wave in the peripheral direction of the shell. After substituting expression (6.14) into Eq. (6.10) and integrated it using Bubnov's method, a second order characteristic equation can be obtained, the roots of which are determined using the formula

$$T_{1,2} = \frac{L}{2\pi R}\left(1+\frac{\pi^2R^2}{\tilde{\varepsilon}L^2}\right) \pm \sqrt{\frac{L^2}{4\pi^2R^2}\left(1+\frac{\pi^2R^2}{\tilde{\varepsilon}L^2}\right)^2-\tilde{\varepsilon}} \tag{6.15}$$

In the case of long shells with parameters $\pi^2R^2/L^2 \ll 1$, the approximate formulae $T_1 \approx L/\pi R$, $T_2 \approx \tilde{\varepsilon}\pi R/L$ follow from this. The smallest of these values is T_2, which we take as the second bifurcation value of the parameter T:

$$T^*_{(2)} = \tilde{\varepsilon}\pi R/L \tag{6.16}$$

We will now show that the value (6.16), which has been found, corresponds completely to the loss of stability of a long cylindrical shell in a purely beam bending mode accompanying its torsion, which was established in the case of a column with a linear axis in Subsection 5.3. To do this, in accordance with the adopted Timoshenko-type model (4.6), we represent the displacements u, v and w in the form

$$u = U+R(\psi\cos\theta-\chi\sin\theta), \quad \upsilon = V\cos\theta-W\sin\theta-R\varphi, \quad w = V\sin\theta+W\cos\theta \tag{6.17}$$

where U, V and W are the displacements of the points of the axial line of the shell along the x, y and z axes of the rectangular Cartesian system of coordinates, and φ, ψ and χ are the angles of rotation about these axes. If it is further assumed that, during the deformation process, a cross-section of the shell x = const remains normal to the axial line Ox after deformation, then the relations of the classical theory of columns $\chi = V'$, $\psi = -W'$ are established between the function χ, ψ and V, W, and, using these relations, we arrive at the representation for the function u

$$u = U-R(W'\cos\theta+V'\sin\theta) \tag{6.18}$$

A prime denotes a derivative with respect to x.

If the external load applied to the shell is conservative and there is no variation in its initial direction, then, according to the results of Section 2, the variational equation

$$\int_0^L\int_0^{2\pi}(S_{11}\delta e_{11}+S_{12}\delta e_{12}+S_{21}\delta e_{21}+S_{22}\delta e_{22}+S_{13}\delta\omega_1+S_{23}\delta\omega_2)Rdxd\theta = 0 \tag{6.19}$$

will hold in the neutral equilibrium state of the shell. In Eq. (6.19), the stresses S_{ij} are determined within the framework of the assumption that $T_{13} = T_{23} = 0$ using formulae (6.3), and the relations

$$\begin{aligned} e_{11} &= U'-R(W''\cos\theta+V''\sin\theta), \quad e_{22} = 0 \\ e_{12} &= V'\cos\theta-W'\sin\theta-R\varphi', \quad e_{21} = W'\sin\theta-V'\cos\theta \\ \omega_1 &= V'\cos\theta+W'\sin\theta, \quad \omega_2 = \varphi \end{aligned} \tag{6.20}$$

are established for the quantities e_{ij} and ω_i using relations (6.17) and (6.18).

Using (6.3) and relations (6.20), starting out from Eq. (6.19) we arrive at the homogeneous differential equation

$$B_{12}R^2\varphi'' = 0$$

which only has a trivial solution and, also, at the system of two homogeneous equations

$$B_{11}RW''' + 2SV'' = 0, \quad -B_{11}RV''' + 2SW'' = 0, \tag{6.21}$$

by which the neutral equilibrium of a thin shell is described according to the classical model of a column.

Eq. (6.21) reduce to a single resolvent of the form

$$B_{11}^2 R^2 V^{\mathrm{IV}} + 4S^2 V'' = 0 \tag{6.22}$$

from which, in the case of a hinged support of the edges $x = 0$ and $x = L$ (within the framework of the theory of columns), the minimum bifurcation value

$$S_{(2)}^* = B_{11}R\pi/(2L) \tag{6.23}$$

follows with an accuracy $1 - \upsilon_{12}\upsilon_{21} \approx 1$, which corresponds to formula (6.16).

Note that, with an accuracy of $t/R + 1 \approx 1$, the result obtained is practically identical to the result established in Subsection 5.3 in the form of formula (5.34) for the case of the twisting of a column by a moment M.

Hence, together with the torsional FLS, which can be realized for low values of the shear modulus G_{12} in the case of shells of arbitrary length and in the case of long shells when $R/L \ll 1$, loss of stability by a purely beam bending mode, which is almost precisely described by functions (6.17) and (6.18), is also possible.

When the equations of membrane theory are used, it is also advisable to consider the case when the number of half-waves in the peripheral direction satisfies the equality $n \geq 2$, that is, when the representation

$$w = W_{1n}\sin(\pi R/L - n\theta) + W_{2n}\cos(\pi R/L - n\theta) \tag{6.24}$$

is taken for the function w.

After substituting expression (6.24) into Eq. (6.10) and integrating it using Bubnov's method, we arrive at the characteristic equation

$$\begin{aligned}&(1-\upsilon_{12}\upsilon_{21})\lambda^3 - \{\lambda^2(\lambda^2+\tilde{\varepsilon}) + (n^2-1)[g_1 n^2 + (1-\upsilon_{12}\upsilon_{21} - 2\upsilon_{12}g_1)\lambda^2]/g_2\}nT + \\ &+ T^2\lambda\{\tilde{\varepsilon}n^2\lambda^2 + (n^2-1)[\lambda^2 + (\tilde{\varepsilon}+g_1)n^2]/g_2\} - \tilde{\varepsilon}n(n^2-1)\lambda^2 T^3/g_2 = 0\end{aligned} \tag{6.25}$$

where $\lambda = \pi R/L$. Since, according to expression (6.16), the estimate $T^* \approx \tilde{\varepsilon}\lambda$ holds in the case of long shells with parameters $\lambda^2 \ll 1$, the formula

$$T^* = \tilde{\varepsilon}\lambda^3/[n^3(n^2-1)] \tag{6.26}$$

follows from Eq. (6.25) with an accuracy $O(\lambda^2)$. This does not have any physical meaning by virtue of the fact that $T^* \to 0$ when $n \to \infty$.

*6.3. Classical solutions of the problem of the stability of cylindrical shells under torsion**

In order to carry out further investigations of the problem, we shall start out from the general equations (6.5), representing the first two of them in the form

$$L_{11}(u) + L_{12}(\upsilon) + \upsilon_{21}\partial_\zeta w = 0, \quad L_{21}(u) + L_{22}^*(\upsilon) + (1 - N^*)\partial_\theta w + T\partial_\zeta w = 0 \tag{6.27}$$

where the operators

$$L_{22}^* = L_{22} + c^2(g_2\partial_{\zeta\zeta} + \partial_{\theta\theta}), \quad N^* = c^2[(\upsilon_{12} + 2g_2)\partial_{\zeta\zeta} + \partial_{\theta\theta}] \tag{6.28}$$

have been introduced. The following differential relations are established from Eq. (6.27):

$$\begin{aligned}D^*(u) &= L_{12}[(1-N^*)\partial_\theta w + T\partial_\zeta w] - \upsilon_{21}L_{22}^*(\partial_\zeta w) \\ D^*(\upsilon) &= \upsilon_{21}L_{21}(\partial_\zeta w) - L_{11}[(1-L^*)\partial_\theta w + T\partial_\zeta w]; \quad D^* = L_{11}L_{22}^* - L_{12}L_{21}\end{aligned} \tag{6.29}$$

* The results in this subsection were obtained jointly with V. A. Ivanov.

Acting with the operator D^* on the third equation of system (6.5) and taking account of relation (6.29), a resolvent of the following form

$$\begin{aligned}&N(w)+c^2g_2^{-1}\{(1+g_1g_2)\partial_{\zeta\zeta}+g_1\partial_{\theta\theta}+2[(\upsilon_{12}+2g_2)\partial_{\zeta\zeta}+\partial_{\theta\theta}]\times\\&\times[(1-\upsilon_{12}\upsilon_{21}-\upsilon_{12}g_1)\partial_{\zeta\zeta}+g_1\partial_{\theta\theta}]\}\partial_{\theta\theta}w+c^2[\tilde{\varepsilon}^{-1}\partial_{\zeta\zeta\zeta\zeta}+2(\upsilon_{12}+2g_2)\partial_{\zeta\zeta\theta\theta}+\partial_{\theta\theta\theta\theta}]\times\\&\times[\partial_{\zeta\zeta\zeta\zeta}+g_2^{-1}(1-\upsilon_{12}\upsilon_{21}-2\upsilon_{12}g_1)\partial_{\zeta\zeta\theta\theta}+\tilde{\varepsilon}\partial_{\zeta\zeta\zeta\zeta}]w\ =\ 0\end{aligned}\tag{6.30}$$

can be obtained with an accuracy $1+c^2\approx 1$. The differential expression for $N(w)$ is given by formula (6.11).

Henceforth, we shall call those solutions of the problem which correspond to the retention in the differential expression (6.11) of the parametric terms obtained using relations (6.3) but without retaining the underlined parameter terms in them, classical solutions. In order to construct such solutions, an analysis of which in the case of isotropic shells has been given in Ref. 19, it is advisable to make further simplifications to Eq. (6.30) which has been derived. For this purpose, we introduce the well-known assumption[18] that the loss of stability of a cylindrical shell under torsion is accompanied by the formation of a small number of half-waves in the axial direction and of a large number of half-waves in the peripheral direction, by virtue of which it is permissible to assume that

$$\partial_{\zeta\zeta}\ll\partial_{\theta\theta}\tag{6.31}$$

Furthermore, we now take into account that the estimate

$$T\approx\partial_{\zeta}\tag{6.32}$$

which corresponds to formula (6.16), holds for the bifurcation value of the parameter T of a long shell.

When account is taken of relations (6.31) and (6.32), Eq. (6.30) can be represented in the simplified form

$$\tilde{\varepsilon}\partial_{\zeta\zeta\zeta\zeta}w+c^2(1+\partial_{\theta\theta})^2\partial_{\theta\theta\theta\theta}w-T(1+\partial_{\theta\theta})\partial_{\zeta\theta\theta\theta}w\ =\ 0\tag{6.33}$$

and, in the case of an isotropic shell, it is identical to the equation derived in Ref. 19 for investigating local forms of loss of stability.

If representation (6.24), in which $L=l$ (l is the length of a dent along the x axis accompanying loss of stability), is introduced into Eq. (6.33) and it is integrated using Bubnov's method, we arrive at the formula ($\lambda_l=\pi R/l$)

$$T^*\ =\ \tilde{\varepsilon}c_0^{3/2}\{n(n^2-1)\mu+1/[\mu^3n^3(n^2-1)]\};\quad \mu\ =\ \sqrt{c_0}/\lambda_l,\quad c_0\ =\ t/(R\sqrt{12\tilde{\varepsilon}})\tag{6.34}$$

for determining of the bifurcation value of the load parameter T.

Unlike formula (6.26), it has a well defined mechanical meaning, since the critical loads for the loss of stability of a shell of arbitrary length with the formation of local dents of length l along the x axis are determined by it.

On minimizing T^* with respect to the parameter μ, we obtain the equality

$$\mu^4\ =\ 3/[n^4(n^2-1)^2]$$

and, using this, we find that

$$T^*\ =\ 4\cdot 3^{-3/4}\tilde{\varepsilon}c_0^{3/2}\sqrt{n^2-1}$$

that is, the smallest non-zero value of T^* is obtained by the transition of a circular section into an oval ($n=2$). In this case,

$$T^*\ =\ T^*_{(3)}\ =\ 0.47\tilde{\varepsilon}^{1/4}(t/R)^{3/2}\tag{6.35}$$

and, for the parameter $\lambda=\pi R/l$ and the length of a dent l, we have the expressions

$$\lambda_l\ =\ \tilde{\varepsilon}^{-1/4}(2t/R)^{1/2},\quad l\ =\ \pi\tilde{\varepsilon}^{1/4}R(R/(2t))^{1/2}$$

It can be shown by direct verification that the results obtained are in complete agreement with assumptions (6.31) and (6.32) and, consequently, with the simplified equations (6.30) which have been adopted. However, to obtain the FLS being investigated, the shell must have a significant length in order that the effect of the boundary conditions on the value of $T^*_{(3)}$ can be neglected.

With the aim of comparing the values of T^* determined using formulae (6.16) and (6.35), we construct the ratio

$$k_1 = T^*_{(3)}/T^*_{(2)} = 0.15\tilde{\varepsilon}^{-3/4}(L/R)(t/R)^{3/2} \tag{6.36}$$

It is seen that satisfaction of the inequality $k_1 > 1$ is possible, in particular, in the case of shells of a sizable relative thickness, that is, in the case of structures which can be classified as columns with respect to their geometrical parameters.

In order to construct another classical solution of the problem, which has been presented earlier[19] in the case of isotropic shells, in addition to condition (6.31) and (6.32) we introduce the assumption

$$\partial_{\theta\theta} \gg 1 \tag{6.37}$$

which enables us to carry out a further simplification of Eq. (6.33) and to represent it in the form

$$\tilde{\varepsilon}\frac{\partial^4 w}{\partial\zeta^4} + c^2\frac{\partial^8 w}{\partial\theta^8} - T\frac{\partial^6 w}{\partial\zeta\partial\theta^5} = 0 \tag{6.38}$$

We shall adopt the representation[18]

$$w = W_{mn}[\sin(\lambda_m\zeta - n\theta) - \sin(\lambda_{m+2}\zeta - n\theta)], \quad \lambda_m = m\pi R/L \tag{6.39}$$

which precisely satisfies the boundary conditions for a hinged mounting

$$w(x=0) = w(x=L) = M_{11}(x=0) = M_{11}(x=L) = 0$$

when the conditions

$$v(x=0) = v(x=L) = 0$$

are imposed.

Substituting expression (6.39) into Eq. (6.38) and equating the factors accompanying the same trigonometrical functions to zero, we obtain the two formulae

$$T = (\tilde{\varepsilon}\lambda_m^4 + c^2n^8)/(\tilde{\varepsilon}\lambda_m^4 + c^2n^8), \quad T = (\tilde{\varepsilon}\lambda_{m+2} + c^2n^8)/(\lambda_{m+2}n^5) \tag{6.40}$$

that is, the equality

$$\lambda_{m+2}(\tilde{\varepsilon}\lambda_m^4 + c^2n^8) = \lambda_m(\tilde{\varepsilon}\lambda_{m+2}^4 + c^2n^8)$$

must be satisfied, which enables us to relate the number of half-waves m along the generatrix of the cylinder to the number n by means of the formula

$$m = (\sqrt{1 + n^8h^2L^4/(16\tilde{\varepsilon}\pi^4R^6)} - 7)/3$$

and, using this, the first formula of (6.40) can be reduced to the form

$$T = \tilde{\varepsilon}\lambda_2^{1/2}c_0^{5/4}(b^4 + a)/(ba^{5/8}); \quad b = m/2, \quad a = c_0^2n^8/\lambda_2^4 \tag{6.41}$$

It was shown in Ref. 19 that a minimum of T is attained for $a = 5.21$ and $b = 0.68$, which leads to the fourth bifurcation value of the loading parameter T

$$T^*_{(4)} = 1.48\tilde{\varepsilon}^{3/8}(R/L)^{1/2}(h/R)^{5/4} \tag{6.42}$$

It should be noted that the result of the minimization of T^* with respect to the parameters m and n leads to the values

$$m = m^*_{(4)} = 1.36, \quad n = n^*_{(4)} = 4.2\tilde{\varepsilon}^{1/8}(R/L)^{1/2}(R/h)^{1/4} \tag{6.43}$$

which correspond to the absolute minimum of the parameter $T^*_{(4)}$. Since these values of $n^*_{(4)}$ and $m^*_{(4)}$ are non-integral, we must seek the minimum of $T^*_{(4)}$ using formula (6.41) for integral values of m and n in the neighbourhood of the values (6.43).

Using formulae (6.16) and (6.42), we construct the ratio

$$k_2 = T^*_{(4)}/T^*_{(2)} = 0.47\tilde{\varepsilon}^{-5/8}(L/R)^{1/2}(t/R)^{5/4} \tag{6.44}$$

which is identical in structure to the ratio given by formula (6.36). Comparing formulae (6.36) and (6.44), it is seen that the inequality $k_1 < k_2$ holds for one and the same relative length of the shell and small value of t/R, while the inequality $k_1 > k_2$ holds for shells of intermediate length.

In conclusion, it should be noted that the solutions constructed correspond to practically the same form of loss of stability and, of these, the first solution holds for long shells and the second for shells of intermediate length or for short shells. Their structure is practically completely determined by the parametric terms in the third equation of system (6.1), which are contained in the expressions for S_{13} and S_{23} and by the instantaneous perturbed state of the shell accompanying loss of stability by the modes which have been investigated.

6.4. Non-classical bending FLS of a shell accompanied by the formation of shallow dents

We take the hinged mounting conditions

$$w = 0, \quad \upsilon = 0, \quad M_{11} = 0, \quad S_{11} = 0 \tag{6.45}$$

at the edges of the shell $x = 0$ and $x = L$ and, of these conditions, the first three will be satisfied if the representations

$$\begin{gathered} w = (w_{nm}\sin n\theta + \tilde{w}_{nm}\cos n\theta)\sin\lambda_m\zeta, \quad \upsilon = (\upsilon_{nm}\sin n\theta + \tilde{\upsilon}_{nm}\cos n\theta)\sin\lambda_m\zeta \\ \lambda_m = m\lambda = m\pi R/L; \quad m = 1, 2, \ldots, \quad n = 1, 2, \ldots \end{gathered} \tag{6.46}$$

are taken for the functions w and υ.

By virtue of the first two conditions, the last condition of (6.45), when relation (6.7) is used, is written in the form

$$\partial_\zeta u + \tilde{\varepsilon}T\partial_\theta u/2 = 0 \text{ When } \zeta = 0, \quad \zeta = L/R \tag{6.47}$$

Adopting the representation

$$u = (U_{nm}\sin n\theta + \tilde{U}_{nm}\cos n\theta)\sin\lambda_m\zeta + (u_{nm}\sin n\theta + \tilde{u}_{nm}\cos n\theta)\cos\lambda_m\zeta$$

for the function u, from conditions (6.47) we establish the relations

$$U_{nm} = \tilde{\varepsilon}nT\tilde{u}_{nm}/(2\lambda_m), \quad \tilde{U}_{nm} = -\tilde{\varepsilon}nTu_{nm}/(2\lambda_m)$$

Consequently,

$$u = \tilde{\varepsilon}nT(-u_{nm}\cos n\theta + \tilde{u}_{nm}\sin n\theta)\sin\lambda_m\zeta/(2\lambda_m) + (u_{nm}\sin n\theta + \tilde{u}_{nm}\cos n\theta)\cos\lambda_m\zeta \tag{6.48}$$

Now, using representations (6.46) and (6.48) we construct the expressions

$$f_i = (f_i^{ss}\sin n\theta + f_i^{cs}\cos n\theta)\sin\lambda_m\zeta + (f_i^{sc}\sin n\theta + f_i^{cc}\cos n\theta)\cos\lambda_m\zeta; \quad i = 1, 2, 3 \tag{6.49}$$

in accordance with system (6.5), where

$$
\begin{aligned}
&f_1^{ss} = -a_1^1\tilde{u}_{nm}, \quad f_1^{cs} = a_1^1 u_{nm}\\
&f_j^{sc} = b_j^1 u_{nm} - b_j^2\tilde{\upsilon}_{nm} + b_j^3 w_{nm}, \quad f_j^{cc} = b_j^1\tilde{u}_{nm} + b_j^2\upsilon_{nm} + b_j^3\tilde{w}_{nm}, \quad j = 1, 2\\
&f_2^{ss} = a_2^1\tilde{u}_{nm} - a_2^2\tilde{u}_{nm} - a_2^2\upsilon_{nm} - a_2^3\tilde{w}_{nm}, \quad f_2^{cs} = -a_2^1 u_{nm} - a_2^2\tilde{\upsilon}_{nm} + a_2^3 w_{nm}\\
&f_3^{ss} = -a_3^1 u_{nm} - a_3^2\tilde{\upsilon}_{nm} + a_3^3 w_{nm}, \quad f_3^{cs} = a_3^1\tilde{u}_{nm} + a_3^2\upsilon_{nm} + a_3^3\tilde{w}_{nm}\\
&f_3^{sc} = b_3^1\tilde{u}_{nm} + b_3^2\upsilon_{nm} + b_3^3\tilde{w}_{nm}, \quad f_3^{cc} = -b_3^1 u_{nm} + b_3^2\tilde{\upsilon}_{nm} + b_3^3 w_{nm}\\
&a_1^1 = \tilde{\varepsilon}nT(g_1 n^2 - \lambda_m^2)/(2\lambda_m^2), \quad b_1^1 = -\lambda_m^2 - g_1 n^2 + \tilde{\varepsilon}^2 n^2 T^2/2\\
&b_1^2 = (\upsilon_{21} + g_1)n\lambda_m, \quad b_1^3 = \upsilon_{21}\lambda_m, \quad a_2^1 = (\upsilon_{12} + g_2)n\lambda_m\\
&a_2^2 = (1 + c^2)(g_2\lambda_m^2 + n^2), \quad a_2^3 = a_3^2 = n\{1 + c^2[(\upsilon_{12} + 2g_2)\lambda_m^2 + n^2]\}\\
&b_2^1 = \varepsilon n^2 T(\upsilon_{12} + g_2)/2, \quad b_2^2 = b_3^3 = nT\lambda_m, \quad b_2^3 = b_3^2 = T\lambda_m\\
&a_3^1 = \upsilon_{12}\lambda_m, \quad a_3^3 = 1 + c^2[\lambda_m^4/\tilde{\varepsilon} + 2(\upsilon_{12} + 2g_2)n^2\lambda_m^2 + n^4], \quad b_3^1 = \tilde{\varepsilon}nT\upsilon_{21}/2
\end{aligned}
$$

The structure of the functions (6.46) and (6.48) indicates that, in the case being considered, it is necessary to put together the system of equations of the Bubnov method in order to find the solutions of Eqs (6.5) in accordance with the approach described earlier in Ref. 20. It consists of the equations

$$
\begin{aligned}
&\int_0^{L/R}\int_0^{2\pi} f_1\left(\cos n\theta\cos\lambda_m\zeta + \frac{\tilde{\varepsilon}nT}{2\lambda_m}\sin n\theta\sin\lambda_m\zeta\right)d\theta d\zeta = 0\\
&\int_0^{L/R}\int_0^{2\pi} f_2\sin n\theta\sin\lambda_m\zeta d\theta d\zeta = 0, \quad \int_0^{L/R}\int_0^{2\pi} f_3\cos n\theta\sin\lambda_m\zeta d\theta d\zeta = 0
\end{aligned}
\tag{6.50}
$$

and the analogous equations which are obtained by replacing $\cos n\theta$ by $\sin n\theta$ and $\sin n\theta$ by $-\cos n\theta$.

Substitution of expressions (6.49) into Eqs (6.50) leads to the system of homogeneous algebraic equations

$$
\begin{aligned}
&\left\lfloor\tilde{\varepsilon}^2 n^2(3 - g_1 n^2/\lambda_m^2)T^2/4 - \lambda_m^2 - g_1 n^2\right\rfloor\tilde{u}_{nm} + (\upsilon_{21} + g_1)n\lambda_m\upsilon_{nm} + \upsilon_{21}\lambda_m\tilde{w}_{nm} = 0\\
&c_{11}\tilde{w}_{nm} + c_{12}\upsilon_{nm} = \upsilon_{12}\lambda_m\tilde{u}_{nm}, \quad c_{12}\tilde{w}_{nm} + c_{22}\upsilon_{nm} = (\upsilon_{12} + g_2)n\lambda_m\tilde{u}_{nm}
\end{aligned}
\tag{6.51}
$$

where

$$
\begin{aligned}
&c_{11} = 1 + c^2\left\lfloor\lambda_m^4/\tilde{\varepsilon} + 2(\upsilon_{12} + 2g_2)n^2\lambda_m^2 + n^4\right\rfloor\\
&c_{22} = (1 + c^2)(g_2\lambda_m^2 + n^2), \quad c_{12} = n + c^2 n[(\upsilon_{12} + 2g_2)\lambda_m^2 + n^2]
\end{aligned}
\tag{6.52}
$$

The relations

$$
\begin{aligned}
&\tilde{w}_{nm} = [\upsilon_{12}\lambda_m c_{22} - (\upsilon_{12} + g_2)n\lambda_m c_{12}]\tilde{u}_{nm}/(c_{22}c_{11} - c_{12}^2)\\
&\upsilon_{nm} = [(\upsilon_{12} + g_2)n\lambda_m c_{11} - \upsilon_{12}\lambda_m c_{12}]\tilde{u}_{nm}/(c_{22}c_{11} - c_{12}^2)
\end{aligned}
\tag{6.53}
$$

are established from the last two equations of system (6.51), and the use of these enables us to determine the bifurcation value of the parameter T

$$
T^* = \sqrt{r_1/r_2}
\tag{6.54}
$$

where

$$
\begin{aligned}
&r_1 = \lambda_m^2 + g_1 n^2 - \upsilon_{21}\lambda_m\left[\frac{\upsilon_{12}\lambda_m c_{22} - (\upsilon_{12} + g_2)n\lambda_m c_{12}}{c_{22}c_{11} - c_{12}^2}\right] -\\
&-(\upsilon_{12} + g_1)n\lambda_m\left(\frac{(\upsilon_{12} + g_2)n\lambda_m c_{11} - \upsilon_{12}\lambda_m c_{12}}{c_{22}c_{11} - c_{12}^2}\right), \quad r_2 = \frac{\tilde{\varepsilon}^2 n^2}{4}\left(3 - \frac{g_1 n^2}{\lambda_m^2}\right)
\end{aligned}
$$

from the first equation.

Table 4

g_1	$R/L = 0.005$			0.01			0.05			0.5		
	$k_3 \times 10^2$	m	n	$k_3 \times 10^2$	m	n	$k_3 \times 10^2$	m	n	$k_3 \times 10^3$	m	n
$t_0 = 0.5$												
0.001	393	55	27	197	55	54	39	11	57	39	1	49
0.01	1242	58	9	621	29	9	127	20	31	124	2	31
0.1	3865	41	2	1932	72	7	386	2	1	386	7	34
$t_0 = 0.2$												
0.001	393	55	27	197	54	53	39	11	54	39	1	49
0.01	124	58	9	921	100	31	124	49	76	124	2	31
0.1	3862	41	2	1931	72	7	386	72	35	386	7	34

It should be noted that the equations obtained from Eqs (6.50) with the above mentioned replacements of $\cos n\theta$ by $\sin n\theta$ and $\sin n\theta$ by $\cos n\theta$ also lead to the result which has been obtained. Consequently, representations (6.46) and (6.48) contain two forms of solutions

$$\begin{aligned} u &= \tilde{u}_{nm}[\cos n\theta\cos\lambda_m\zeta + \tilde{\varepsilon}nT\sin n\theta\sin\lambda_m\zeta/(2\lambda_m)] \\ \upsilon &= \upsilon_{nm}\sin n\theta\sin\lambda_m\zeta, \quad w = \tilde{w}_{nm}\cos n\theta\sin\lambda_m\zeta \end{aligned} \tag{6.55}$$

$$\begin{aligned} u &= u_{nm}[\sin n\theta\cos\lambda_m\zeta - \tilde{\varepsilon}nT\cos n\theta\sin\lambda_m\zeta/(2\lambda_m)] \\ \upsilon &= \tilde{\upsilon}_{nm}\cos n\theta\sin\lambda_m\zeta, \quad w = w_{nm}\sin n\theta\sin\lambda_m\zeta \end{aligned} \tag{6.56}$$

which lead to the same formula (6.54). These solutions are non-classical in the case of the problem of the stability of a cylindrical shell under torsion since only the underlined parametric terms in relations (6.3) participate in the derivation of formula (6.54) while only the parametric terms, contained in the expressions for S_{13} and S_{23}, participate in the construction of the classical solutions presented in the preceding subsection.

Hence, in the case of the torsion of a cylindrical shell, loss of stability according to a fifth form, described by the functions (6.55) and (6.56) is also possible, which is a bending mode by virtue of the conditions$w \neq 0$, $\upsilon \neq 0$, $u \neq 0$. When it occurs, the minimum bifurcation value of the parameter $T^*_{(5)}$ is determined using formula (6.54) by minimizing it with respect to the numbers m and n. The results of such calculations in the form of the ratio $T^*_{(5)}/T^*_{(2)} = k_3$ and the corresponding critical numbers m and n are shown in Table 4 for two values of the parameter $t_0 = t/R$ when $\tilde{\varepsilon} = 1$.

An analysis of the results obtained enables us to formulate the following main conclusions.

1°. The value of the parameter $T^*_{(5)}$ depends strongly on the parameters R/L and G_{12}/E_1 and is practically independent of the thickness of the shell, while the critical loads, corresponding to a classical FLS, are independent of the parameter G_{12}/E_1 for the degree of accuracy in their determination which has been assumed.

2°. The realization of loss of stability according to the FLS established in this section, earlier than according to the other forms, is possible in the case of short shells and shells of intermediate length with of a sizeable relative thickness t_0 for of a small value of the parameter G_{12}/E_1, which is accompanied by the formation of a large number of shallow dents in the peripheral direction of the shell.

3°. In the case of long shells (when $R/L \leq 0.01$), $k_3 > 1$ for all values of G_{12}/E_1. The value of this coefficient decreases as the parameter G_{12}/E_1 decreases.

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